CS 6815 Pseudorandomness and Combinatorial Constructions **Fall 2019**

Lecture 8: September 24th

Lecturer: Eshan Chattopadhyay *Scribe: Rishi Advani*

8.1 Expanders - Explicit constructions

In this lecture, we formulate ways to take products of graphs to construct larger expander graphs.

New notation: Let a (N, D, γ)-graph be a (N, γ) spectral expander that is D-regular.

Our approach: start with small graphs and iteratively construct larger graphs.

8.2 Squaring Graphs

Intuitively, squaring is just 2 hops on the original graph. Note that self loops and multiple edges are allowed in squared graphs.

Formally, if we have a graph $G = (V, E)$, let $G^2 = (V, E')$ be a graph such that, for all v in V, the (i, j)th neighbor of v is the jth neighbor of the ith neighbor of v, where $i, j \in [D]$ (are numbers from 1 to D).

This operation doesn't add any nodes, and it squares the number of edges. A^2 is the normalized adjacency/random-walk matrix of G^2. Hence, $\lambda(G^2) = \lambda(G)^2$.

- degree increases :(
- nodes remain same :/
- expansion improves :)

8.3 Tensor Products

For V in R_n, W in R_m, the tensor product of V and W is denoted as $Z = V \otimes W \in R_n m$. It is a generalization of the outer product.

For two vectors, we define their tensor product to be a matrix, such that $z_{ij} = v_i w_j, i \in [n], j \in [m]$. For two matrices $A \in R_{n_1} \times R_{n_2}$, $B \in R_{m_1} \times R_{m_2}$, the entries of the tensor product $C = A \otimes B$ are as follows:

$C_{i_1 i_2 j_1 j_2} = A_{i_1 j_1} B_{i_2 j_2}$

Some properties of tensor products:

1. $A \otimes (B + C) = A \otimes B + A \otimes C$
2. in general, $A \otimes B \neq B \otimes A$
3. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ if AC and BD are defined by the standard rules of matrix multiplication

4. $(A \otimes B)^T = A^T \otimes B^T$

With property number 3, if $A_{n \times n}$ and $B_{m \times m}$ are matrices and C_n and D_m are vectors, A effectively acts just on C and B just on D. This is a major part of the intuition for why tensor products can be useful.

8.4 Tensor Products of Graphs

Suppose we have G_1, G_2, such that:
G_1 is an (N_1, D_1, γ_1)-graph, and its adjacency matrix is M_1.
G_2 is an (N_2, D_2, γ_2)-graph, and its adjacency matrix is M_2.

Then, we define the tensor product of G_1 and G_2 to be $G = G_1 \otimes G_2$.

The adjacency matrix of G is $M_1 \otimes M_2$. The set of vertices of G is $[N_1] \times [N_2]$. (v, j) is a neighbor of (u, i) if (u, v) is in E_{G_1}, (i, j) in E_{G_2}.

To visualize this, make 4 "clouds" that are copies of the vertices of G_2; each cloud represents one vertex of G_1. Draw an edge between two vertices in different clouds if the vertices corresponding to the clouds in G_1 are connected, and the vertices corresponding to the positions in the intra-cloud graph are connected in G_2.

Now, we analyze the spectral expansion of G.

The eigenvalues of $A_1 \otimes A_2$ are $\lambda_i(G_1)\lambda_j(G_2), i \in [N_1], j \in [N_2]$ – the largest eigenvalue is $1 \cdot 1$, so the second largest is $1 \cdot \lambda_{G_1}$ or $1 \cdot \lambda_{G_2}$.

G is $(N_1N_2, D_1D_2, \min(\gamma_{G_1}, \gamma_{G_2}))$.

- degree increases :(
- nodes increase :)
- expansion remains same :/

There is a more inituive proof of the spectral expansion for tensor products that helps build the intuition needed to think about the zig-zag product. The rest of this scribed document will be focused on this proof.

8.4.1 Intuitive Proof of Spectral Expansion for Tensor Products

$A = A_1 \otimes A_2$

w.t.s. that $||Ax|| \leq \lambda ||x||, x \perp 1_{N_1N_2}$

x is a long vector, but we'll think of it as the flattened out form of a matrix that is $N_1 \times N_2$. Think of x as a probability distribution; the ith row is the marginal of x on the ith cloud.

Write x as $x^{\parallel} + x^{\perp}$, where $x^{\parallel}$ is parallel to u_{N_2} (where u is the normalized all-ones vector) on each cloud. Visualize $x^{\parallel}$ and $x^{\perp}$ as matrices of the same dimension as x.

$x^{\parallel} = y \otimes u_{N_2}$, for some unique vector y in R^{N_1}. Note that y is perpendicular to u_{N_1}.

$Ax^{\parallel} = (A_1 \otimes A_2)(y \otimes u_{N_2}) = (A_1 y \otimes A_2 u_{N_2})$

u_{N_2} is an eigenvector with eigenvalue 1.

$||Ax^{\parallel}|| = ||A_1 y|| \cdot ||u_{N_2}||$ (operator norm is multiplicative on tensor product)

The matrix shrinks the L_2 norm of the vector by its second largest eigenvalue, so we have $\lambda_{G_1}||y|| \cdot ||u_{N_2}|| = \lambda_{G_1}||x^{\parallel}||$

Now we consider $||Ax^{\perp}||$.

Each row of $x^{\perp}$ is perpendicular to the all-ones vector. If A_2 acts on $x^{\perp}$ it wil shrink each row by λ (i.e. $||A_2(x^{\perp})_1|| \leq \lambda_{G_2}||(x^{\perp})_1||$).

$||Ax^{\perp}|| = (A_1 \otimes A_2)x^{\perp} = (A_1 \otimes I_{N_2})(I_{N_1} \otimes A_2)x^{\perp}$ because the matrices are of right dimension, so we can use the tensor property that we discussed earlier.

How does $(I_{N_1} \otimes A_2)$ act on $x^{\perp}$? Each row will be A_2 times the corresponding row. It shrinks each row of $x^{\perp}$ by λ_{G_2}.

$||Ax^{\perp}|| = \lambda_{G_2}||A_1|| \cdot ||x^{\perp}|| \leq 1 \leq \lambda_{G_2}||x^{\perp}||$

We have finished both the calculations, so we will finish the proof now. In one term we get λ_{G_1}, and in the other we get λ_{G_2}.

$||Ax||$ is equal to $||A(x^{\parallel} + x^{\perp})||$. By the triangle inequality, $||A(x^{\parallel} + x^{\perp})|| \leq ||Ax^{\parallel}|| + ||Ax^{\perp}|| \leq \lambda_{G_2}||x^{\parallel}|| + \lambda_{G_1}||x^{\perp}|| \leq (\lambda_{G_1} + \lambda_{G_2})||x||$.

But this is a worse bound then we promised. We promised max, not sum. In order to get a better bound, we observe that, if we can show that $Ax^{\parallel}$ and $Ax^{\perp}$ are orthogonal vectors, we can use the Pythagorean Theorem instead of the triangle inequality to get a stronger bound.

Claim 8.1 *$Ax^{\parallel}$ and $Ax^{\perp}$ are orthogonal vectors.*

Proof:

$Ax^{\perp}$ is perpendicular to u_{N_2} on each cloud because, in the expression $(A_1 \otimes I_{N_2})(I_{N_1} \otimes A_2)x^{\perp}$, the application of A_2 keeps the vector perpendicular, and the application of A_1 replaces each cloud with a linear combination of clouds, which also preserves the orthogonality.

$Ax^{\parallel}$ remains parallel to u_{N_2} on each cloud because $x^{\parallel} = y \otimes u_{N_2} Ax^{\parallel} = (A_1 \otimes A_2)(y \otimes u_{N_2}) = (A_1 y \otimes u_{N_2})$.

Thus, $Ax^{\parallel}$ and $Ax^{\perp}$ are orthogonal vectors. ■

We can now give the desired stronger bound using the orthogonality of the two vectors:

$||Ax||^2 = ||Ax^{\perp}||^2 + |Ax^{\parallel}||^2 \leq \lambda_{G_2}^2||x^{\perp}||^2 + \lambda_{G_1}^2||x^{\parallel}||^2 \leq \max\{\lambda_{G_1}, \lambda_{G_2}\}^2(||x^{\parallel}||^2 + |x^{\perp}||^2) = \max\{\lambda_{G_1}, \lambda_{G_2}\}^2||x||^2$

$||Ax|| \leq \max\{\lambda_{G_1}, \lambda_{G_2}\}||x||$

8.5 Concluding Remarks

We will cover the zigzag product in the next class.