CS 6815 Pseudorandomness and Combinatorial Constructions **Fall 2019**

Lecture 1: September 17

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1.1 Introduction

Suppose $G = (V, E)$ is a D-regular graph with $\gamma = 1 - \lambda$ spectral expansion, where $|V| = N$. Then a random walk P of length t is chosen as follows:

- randomly pick the first vertex v_1;
- randomly pick a neighbor of last picked vertex for $t - 1$ steps
- generate a random walk $P : l_1 \to l_2 \to ... \to l_t$

Theorem 1.1 (Hitting Property of Expander Walks) *For any set $B \subset V$,*

$$\Pr[\textit{Random walk } P \textit{ stays in } B] \le (\mu_B + (1 - \mu_B)\lambda)^t$$

where $\mu_B = \frac{|B|}{N}$ is the density of set B.

1.2 Notation and Preliminary

Throughout this lecture we are going to use following notation:

- $\mathbf{1}$ denotes the vector of all 1's: $\mathbf{1} = (1, ..., 1)$.
- $J \in \mathbb{R}^{N \times N}$ denotes the matrix with all entries equal to $1/N$.
- $\mathbf{l}_B$ denotes the indicator vector of set $B : j \in B \iff (\mathbf{l}_B)_j = 1$.
- $\mathbf{u} \in \mathbb{R}^N = \frac{1}{N}\mathbf{1} = (\frac{1}{N}, ..., \frac{1}{N})$.
- $< x, y >$ denotes the inner product of x and y.
- A is the normalized adjacency matrix of G.

1.2.1 Spectral Norm of Matrices

Definition 1.2 *Let $x \in \mathbb{R}^n$, the p-norm of x is defined as*

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}, p \ge 1$$

Definition 1.3 *Let $M \in \mathbb{R}^{n\times n}$, the 2-norm of M is defined as:*

$$\|M\|_2 = \max_{x\in\mathbb{R}^n} \frac{\|Mx\|_2}{\|x\|_2} = \max_{x\in\mathbb{R}^n, \|x\|_2=1} \|Mx\|_2$$

Property 1.4 *2-norm of matrices satisfies the following properties:*

1. *If M is a symmetric matrix, then $\|M\|_2 = |\lambda_1|$ where λ_1 is the largest eigenvalue of M.*
2. $\|M_1 + M_2\|_2 \leq \|M_1\|_2 + \|M_2\|_2$.
3. $\|M_1 M_2\|_2 \leq \|M_1\|_2 \|M_2\|_2$.

1.2.2 Vector and matrix decomposition

Lemma 1.5 (Vector decomposition) $\forall x \in \mathbb{R}^n, x = x^{\|} + x^{\perp}$*, where* $x^{\|} = < x, \mathbf{1} > \cdot \mathbf{1}, x^{\perp} = x - x^{\|}$

Lemma 1.6 (Matrix decomposition) $A = \gamma J + \lambda E$*, then* $\|E\|_2 \leq 1$

Proof: Define $E = \frac{A-\gamma J}{\lambda}$. Let $x \in \mathbb{R}^n, \|x\|_2 = 1 . x = x^{\|} + x^{\perp}$. Then $Ax^{\|} = x^{\|}, Jx^{\|} = x^{\|}, Jx^{\perp} = 0$. Hence

$$\begin{aligned}
\lambda Ex &= (A - \gamma J)(x^{\|} + x^{\perp}) = (1-\gamma)x^{\|} + Ax^{\perp} \\
\Rightarrow Ex &= x^{\|} + \frac{1}{\lambda} Ax^{\perp} \\
\Rightarrow \|Ex\|_2^2 &\leq \|x\|_2^2 + \frac{1}{\lambda^2}\|Ax^{\perp}\|_2^2 \leq \|x^{\|}\|_2^2 + \lambda^2 \|x^{\perp}\|_2^2 \\
&\leq \|x^{\|}\|_2^2 + \|x^{\perp}\|_2^2 = \|x\|_2^2 = 1
\end{aligned}$$

Thus $\|E\|_2 \leq 1$. ■

1.3 Proof of Theorem 1.1

Proof:[Theorem 1.1]

Claim 1.7

$$\Pr[\textit{Random walk } P \textit{ stays in } B] = \|\mathbf{u}^T D_B (D_B^T A D_B)^{t-1}\|_1$$

Proof of:[Claim1.7] The equality follows by induction on t. □

According to Claim 1.7, we have:

$$\begin{aligned}
&\Pr[\text{Random walk } P \text{ stays in } B] = \|\mathbf{u}^T D_B (D_B A D_B)^{t-1}\|_1 \\
\leq &\sqrt{|B|}\|\mathbf{u}^T D_B (D_B A D_B)^{t-1}\|_2 \\
\leq &\sqrt{|B|}\|\mathbf{u}^T D_B\|_2 \|D_B A D_B\|_2^{t-1} \\
\leq &\sqrt{|B|} \cdot \frac{1}{N} \cdot \|\mathbf{l}_B\|_2 \|D_B A D_B\|_2^{t-1} \\
= &\mu_B \|D_B A D_B\|_2^{t-1}
\end{aligned}$$

Notice that $\|D_B A D_B\|_2^{t-1}$ can be written as $\|D_B A D_B\|_2 = \gamma \|D_B J D_B\|_2 + \lambda \|D_B E D_B\|_2$

Claim 1.8 $\|D_B E D_B\|_2 \le 1$

Proof of:[Claim1.8] $\|D_B E D_B\|_2 \le \|D_B\|_2^2 \|E\|_2 \le 1$. □

Claim 1.9 $\|D_B J D_B\|_2 \le \mu_B$

Proof of:[Claim1.9] Let $x \in \mathbb{R}^N, \|x\|_2 = 1$, let $x_B = D_B x$. Then

$$
\begin{aligned}
Jx_B &= \frac{1}{N} \sum_{i \in B} x_i \cdot \mathbf{1}, \\
D_B J x_B &= \frac{1}{N} \sum_{i \in B} x_i \mathbf{1}_B, \\
\|D_B J D_B\|_2 &\le \frac{1}{N} \left| \sum x_i \right| \sqrt{B} \\
&\le \frac{1}{N} (\sum x_i^2) \sqrt{B} \sqrt{B} \le \frac{|B|}{N} = \mu_B \qquad \square
\end{aligned}
$$

Hence $\|D_B A D_B\|_2 = \gamma \|D_B J D_B\|_2 + \lambda \|D_B E D_B\|_2 \le (1-\lambda)\mu_B + \lambda = \mu_B + (1-\mu_B)\lambda$.

Thus, we have

$$\Pr[\text{Random walk } P \text{ stays in } B] = \|\mathbf{u}^T D_B (D_B^T A D_B)^{t-1}\|_1 \le (\mu_B + (1-\mu_B)\lambda)^t.$$

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