**CS 6815 Pseudorandomness and Combinatorial Constructions** **Fall 2019**

# Lecture 4: September 10

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## 4.1 Overview

Today, we cover two definitions of **Expander Graphs**, which can be loosely described as "sparse, well-connected graphs," as well as introduce the statement of the **Expander Mixing Lemma**.

## 4.2 Vertex Expander

**Definition 4.1** *Let G = (V, E) be an undirected, D-regular graph with N vertices. G is a (k, A) expander if* $\forall S \subseteq [N], |S| \leq k, |N(S)| \geq A(S)$, *we have that* $N(S) = \{v \in [N] | \exists e = \{u, v\}$ *for some* $u \in [N]\}$.

*We will think of each undirected edge* $\{u, v\}$ *as two directed edges* $(u, v)$ *and* $(v, u)$.

*The probabilistic method can be used to prove the following: (we saw a proof of this in class)*

***Lemma 4.2*** *There are expander graphs with D = O(1) i.e. very sparse,* $k = \Omega(N)$ *i.e. of constant density,* $A = \Omega(1)$ *i.e. limited by a constant (not the number of vertices).*

## 4.3 Preliminaries

### 4.3.1 Linear Algebra Refresher

***Theorem 4.3 (Spectral Theorem for Symmetric Matrices)*** *If* $M \in \mathbb{R}^{n \times m}$ *is symmetric,* $\exists$ *orthonormal vectors,* $v_1, ..., v_n$ *(the* **eigenvectors** *of M) and real numbers* $\lambda_1 \geq ... \geq \lambda_n$ *(the* **eigenvalues** *of M) such that* $Mv_i = \lambda_i v_i \ \forall i \in [n]$.

***Definition 4.4 (Rayleigh Quotient)*** *For* $M \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$, *we define* $R(M, x) \equiv \frac{x^\top Mx}{x^\top x}$.

***Claim 4.5*** $\max_{x \in \mathbb{R}} R(M, x) = \lambda_1$.

**Proof:** Let $\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$. We see that

$$\begin{aligned} x^\top M x &= \left(\sum_{i=1}^{n} \alpha_i \vec{v}_i\right)^\top M \left(\sum_{i=1}^{n} \alpha_i \vec{v}_i\right) \\ &= \left(\sum_{i=1}^{n} \alpha_i \vec{v}_i\right)^\top \left(\sum_{i=1}^{n} \lambda_i \alpha_i \vec{v}_i\right) \\ &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i. \end{aligned}$$

From this, we can conclude that $x^\top M x \leq \lambda_1 (\sum_{i=1}^{n} \alpha_i^2) \leq \lambda_1$.

***Claim 4.6*** $\max_{x \perp \text{ span}(\vec{v}_i, \ldots, \vec{v}_{i-1})} R(M, x) = \lambda_i$. In particular, $\lambda_2 = \max_{x \in |\mathbb{R}^n|, |\lambda_1|_2 = 1, x \perp v_1} R(M, x)$.

### 4.3.2 Notation

Let $A$ be a normalized adjacency matrix of $G$, i.e. $A(i, j) = \begin{cases} \frac{1}{D} & (i,\, j) \in E \\ 0 & otherwise \end{cases}$

***Claim 4.7*** *(1)* $\lambda_1 = 1$*, (2)* $\lambda_n \geq -1$*, (3)* $\lambda_2 = 1$ *if and only if* $G$ *is disconnected.*

**Proof:**

1. $A\vec{1} = 1 \cdot \vec{1} \rightarrow \lambda_1 \geq 1$
   $\sum_{i<j} (x_i - x_j)^2 A_{ij} = x^\top x - x^\top A x$.
   We rearrange to get $x^\top A x = x^\top x - \sum_{i<j} (x_i - x_j)^2 A_{ij} \rightarrow |x^\top A x| \leq x^\top x \rightarrow R(A, x) \leq 1 \forall x \in \mathbb{R}^n \rightarrow \lambda_1 \leq 1$.
2. Left as an exercise to the reader.
3. Assume $||x||_2 = 1$ and $x^\top A x = 1 - \sum_{i<j} (x_i - x_j)^2 A_{ij}$. Recall that $\lambda_2 = \max_{x \in \mathbb{R}^n, ||x||_2 = 1, x \perp \vec{1}} x^\top A x$. This is equivalent to $\lambda_2 = 1 - \inf_{x \in \mathbb{R}^n, ||x||_2 = 1, x \perp \vec{1}} \sum_{i<j} (x_i - x_j)^2 A_{ij}$. We know that $\sum_{i<j} (x_i - x_j)^2 A_{ij} = x^\top x - x^\top A x = 0$, since $x$ is normalized, making $x^\top x = 0$, and setting $Ax = \lambda_2 x$. Note that $\sum_{i<j} (x_i - x_j)^2 A_{ij}$ is known as the *Laplacian* of a matrix. We observe that while there must exist positive and negative entries, as the Euclidean norm of $x$ is 1, there cannot exist any edge between coordinates of different signs ($x_i = x_j$ for nonzero $A_{ij}$), as this would mean that the Laplacian would be nonzero. Therefore, the graph has at least two components, and is not connected. The other direction of the proof is left as an exercise to the reader.

## 4.4 Spectral Expander

***Definition 4.8*** *Let* $G = (V,\, E)$ *be an undirected, D-regular graph with N vertices. Recall that* $\lambda(G) (\equiv \max\{|\lambda_2|, |\lambda_N|\})$*. G is a (N,* $\alpha$*) spectral expander if* $\lambda(G) = \leq 1 - \alpha$*.*

## 4.5 Expander Mixing Lemma

*Let be the second largest eigenvalue of the normalized adjacency matrix of the graph G. For any two subsets of vertices, let* $E(S, T)$ *denote the number of edges between* $S$ *and* $T$*. Thus, recalling that we think of undirected* *edges as two directed edges, $E(S,T)$ counts each edge between $S$ and $T$ twice.*

***Lemma 4.9*** Let $G = (N, \alpha)$ be a spectral expander. $\forall S, T \subseteq [N]$, we have

$$\left| \frac{E(S,T)}{ND} - \alpha\beta \right| \leq \lambda\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$$

with $\frac{|S|}{N} = \alpha$, $\frac{|T|}{N} = \beta$. Note that the last two terms under the square root may be omitted for sufficiently small $\alpha, \beta$ (or equivalently, for sufficiently large $\frac{|S|}{N}, \frac{|T|}{N}$). Further, note that we do not require $S$ and $T$ to be disjoint.