CS 6810: Theory of Computing **Fall 2023**

Lecture 3: Aug 29, 2021

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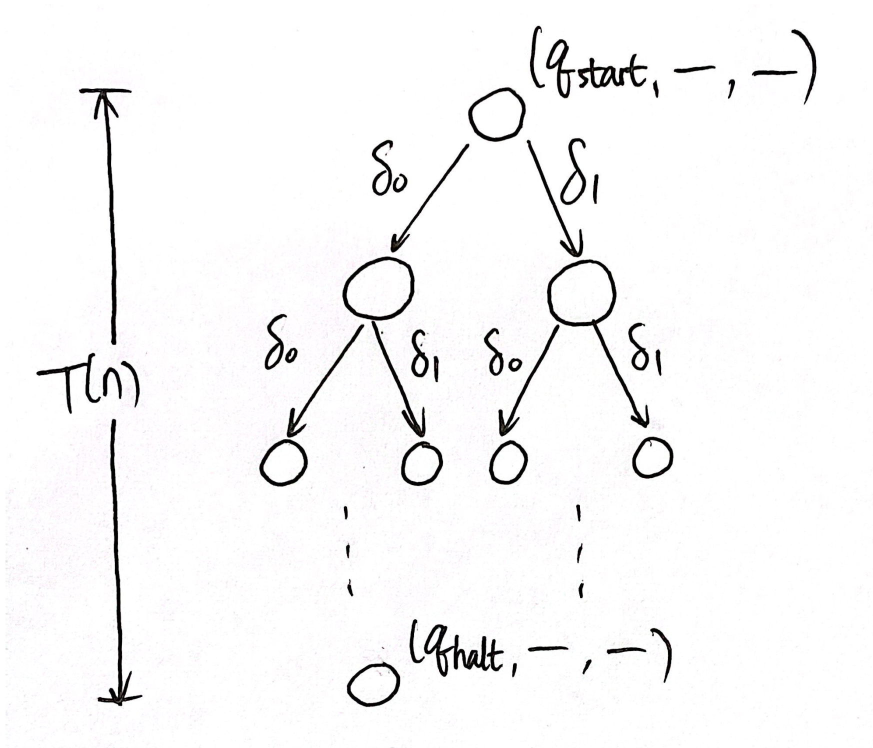
1 Two Equivalent Definitions of NP

Recall: Non-deterministic Turning Machines (NDTMs)

The configuration of a Turing Machine (TM) includes the current state, contents of non-blank work tape cells, and head locations.

In the following diagram, each circle represents a configuration of a NDTM:



The transition functions are $\delta_i : Q \times \Gamma^{k+2} \to Q \times \Gamma^{k+1} \times \{L, R, S\}^{k+2}$. x is accepted by the NDTM N if there exists a sequence of δ_0 and δ_1's such that N halts and writes 1 on the output tape when it follows that sequence.

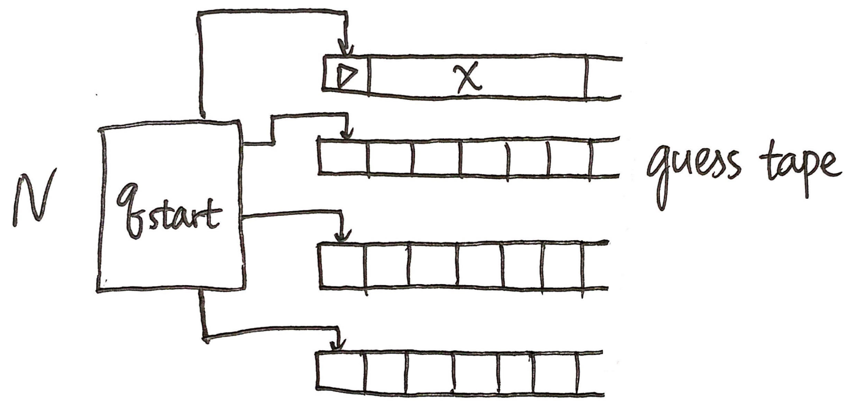
We say that the NDTM **runs in time** $\mathbf{T(n)}$ if the depth of the computation tree is $T(n)$. Note that the number of possible paths is exponential in $T(n)$.

Definition 1.1. $L \in \mathsf{NP}$ *if there exists a polynomial time verifier* V *such that for all* $x \in \{0,1\}^*$, $x \in L$ *if and only if there exists some* $y \in \{0,1\}^{c|x|^c}$ *such that* $V(x, y) = 1$.

Definition 1.2. $L \in \mathsf{NP}$ *if there exists a polynomial time NDTM* N *that computes* L.

Claim 1.3. *The two definitions of* NP *are equivalent.*

Proof. First suppose that $L \in \mathsf{NP}$ in the sense of Definition 1.1. Then we have a verifier V such that $x \in L \Leftrightarrow \exists y \in \{0,1\}^{c|x|^c}, V(x,y) = 1$. The way we construct N is by "non-deterministically guessing y and simulating $V(x,y)$". More specifically, consider the following NDTM:



After computing $|x|$, the length of the input, the NDTM writes a guess of y on the guess tape in the first $c|x|^c$ steps. Then, it simulates V on (x,y) and outputs according to V. The correctness and the runtime of the NDTM are straightforward.

Next, suppose that $L \in \mathsf{NP}$ in the sense of Definition 1.2. Then by assumption, we have a NDTM N that runs in cn^c time and computes L. The verifier V is constructed as follows: On input x and certificate $y \in \{0,1\}^{c|x|^c}$, simulate N on x using the ith bit of y to choose δ_0 or δ_1. $\square$

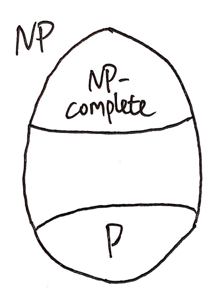
2 Reductions and NP-completeness

Definition 2.1. *We say that L_1 is poly-time Karp reducible to L_2 ($L_1 \leq_P L_2$) if there exists a polynomial time TM M such that $x \in L_1 \Leftrightarrow M(x) \in L_2$.*

Definition 2.2. *L is NP-**complete** if $L \in \mathsf{NP}$ and for all $L' \in \mathsf{NP}, L' \leq_P L$.*

Example 2.3. *Some examples of NP-complete problems: SAT, 3SAT (Cook-Levin), 3-Coloring, TSP.*

Many computer scientists believe that $\mathsf{P} \neq \mathsf{NP}$:



We'll prove in the next lecture that if $\mathsf{P} \neq \mathsf{NP}$, then there exists a problem in NP that is not in P and is not NP-complete.

3 Time Hierarchy Theorem

Definition 3.1. *$T(n)$ is **time-constructible** if a TM on input 1^n can write $1^{T(n)}$ in $cT(n)$ time and $T(n) \geq n$.*

Example 3.2. *Examples of time-constructible functions include $2^n, 2^{2^n}$, and $2^{2^{\sqrt{\log n}}}$.*

Exercise 3.3. *Construct a function T satisfying $T(n) \geq n$ that is not time-constructible.*

Theorem 3.4. *Suppose f and g are time-constructible functions from $\mathbb{N}$ to $\mathbb{N}$, and $f(n) \cdot \log f(n) = o(g(n))$. Then $\mathsf{DTIME}(f(n)) \subsetneq \mathsf{DTIME}(g(n))$.*

Proof. Consider the following TM D: On input $w = (x, y)$, simulate M_x on w for $g(|w|)$ simulation steps. If M_x halts at some point, flip the output. Otherwise, output 0.

By construction, $L(D) \in \mathsf{DTIME}(g(n))$. Now let x^* be such that M_{x^*} runs in $cf(n)$ time. Then we claim that $L(M_{x^*}) \neq L(D)$. In other words, there exists a w such that $M_{x^*}(w) \neq D(w)$.

To prove the claim, notice that M_{x^*} on input $w = (x^*, 1^k)$ can be simulated in $C' \cdot f(|w|) \cdot \log(f(|w|))$ steps. This is less than $g(|w|)$ for large enough k. Therefore, M_{x^*} halts on w in less than $g(|w|)$ steps, so by the definition of D, $M_{x^*}(w) \neq D(w)$. This shows that $\mathsf{DTIME}(f(n)) \subsetneq \mathsf{DTIME}(g(n))$. □

Remark 3.5. *Using 1^k as a part of the input is called the "padding argument". It is useful here because the description of the TM M_{x^*} may not be long enough for the asymptotic bound $f(n) \cdot \log f(n) = o(g(n))$ to apply.*

Remark 3.6. *This argument does not generalize directly to NDTMs because we can't flip the output of NDTMs.*