CS 6810: Theory of Computing **Fall 2023**

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1 Design Construction for NW-PRG

Recall that last lecture we proved the following theorem with the assumption that some (n, k)-design exists.

Theorem 1.1. *Suppose $L \in DTIME(T(n))$ is (S, ϵ)-hard, where L is a series of (S, ϵ)-hard functions for every input length. Then there exists (S', ϵ')-PRGs $\{G_n : \{0,1\}^{r(n)} \to \{0,1\}^{m(n)}\}_n$ computable in time $m(n) \cdot T(n)$, where $s' = s - O(m \cdot 2^k)$ and $\epsilon' = m \cdot \epsilon$.*

Definition 1.2. *A (n, k)-design in universe $[r]$ is a collection of sets $S_1, S_2, \ldots, S_m \subseteq [r]$ where $|S_i| = n$ and $\forall i \neq j$, $|S_i \cap S_j| \leq k$.*

We now show how to construct the design and how to set the parameters. First, fix field $\mathbb{F} = \mathbb{F}_n$. For each polynomial $P(x)$ on $\mathbb{F}$ of degree $\leq k$, set $S_p\{(a, p(a)) : a \in \mathbb{F}\}$.

Claim 1.3. *For $p \neq q$, $|S_p \cap S_q| \leq k$*

The claim is proven by realizing that $p - q$ has at most k roots, since they are polynomials of degree $\leq k$. Therefore, p and q intersect at most k times. With this construction, $r = n^2$ because both a and $p(a)$ range from 1 to n, and $m = n^{k+1}$.

Next, fix small constant $\delta > 0$, so that $S = 2^{\delta n}$ and $\epsilon = 2^{-\delta n}$. Can we take $T(n) = poly(n)$? In other words, is there a language $L \in DTIME(poly(n))$ that is $(2^{\delta n}, 2^{-\delta n})$-hard? We know that $P \subset P/poly$, so there are no languages in P that cannot be approximated by an exponential sized circuit. Instead, we assume $T(n) = 2^{O(n)}$. i.e. $L \in E$.

Now, set $k = \frac{n}{10 \log(n)}$. It follows that $m = n^{k+1} = 2^{\frac{\delta n}{10}} = 2^{\frac{\delta}{10}\sqrt{r}}$.

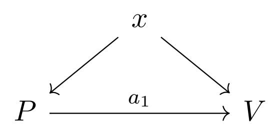
To conclude, if we are given a $L \in E$ that is $(2^{\delta n}, 2^{-\delta n})$-hard, then we can produce a (S', ϵ')-PRG that takes in $r = n^2$ bits and outputs exponential bits, where $S' = 2^{\delta n} - 2^{\frac{\delta}{10}n} = \Omega(2^{\delta n})$ and $\epsilon' = 2^{\frac{\delta}{10}n} 2^{-\delta n} \leq 2^{-\frac{\delta}{2}n}$. This implies any algorithm in BPP can be derandomized to a deterministic algorithm that runs in time $n^{O(\log n)}$. We note that one can construct better designs that will yield BPP=P under the circuit lower bound assumptions against E.

We now switch topics to a new type of proof system.

2 Interactive Proofs

2.1 NP

Now, let us revisit the setting of NP but through the lens of the verifier. Recall that NP is the set of decision problems verifiable by a deterministic Turing machine in polynomial time. An alternative way of viewing NP is as a simple interactive proof system, with a "prover" in addition to the verifier. On input x, the all-powerful prover P comes up with some certificate for x, a_1, and sends it to the verifier V.



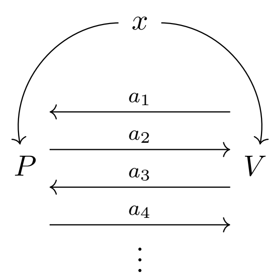
If $x \in L$, then some prover that can convince the verifier that x is in L. If $x \notin L$, then no prover can convince the verifer that x is in L. Formally, for $L \in NP, \exists V$ such that:

$$x \in L \;\Rightarrow\; \exists P \; s.t. \; V(x, a_1) = 1$$
$$x \notin L \;\Rightarrow\; \forall P, \; V(x, a_1) = 0$$

2.2 Deterministic Interactive Proofs (dIP)

Now, let us define a proof system by extending the number of rounds of communication.

Definition 2.1. *Define a complexity class $dIP[k(n)]$ where n is the length of x, and $k(n)$ is the number of rounds of communication between the prover and verifier.*



[Note that all messages $a_1, a_2, \ldots, a_{k(n)}$ between prover and verifier are polynomial in the size of x]. Let $out_V\langle V, P\rangle(x)$ stand for the final output of the verifier. We now have: For $L \in dIP[k(n)], \exists V$ such that:

$$x \in L \;\Rightarrow\; \exists P \; s.t. \; out_V\langle V, P\rangle(x) = 1$$
$$x \notin L \;\Rightarrow\; \forall P, \; out_V\langle V, P\rangle(x) = 0$$

Definition 2.2. $dIP = \bigcup_{c \in \mathbb{N}} dIP[n^c]$

Claim 2.3. *$dIP = NP$ (Nothing is gained by allowing communication.)*

Proof:

$NP \subseteq dIP$ because $NP = dIP[1]$.

$dIP \subseteq NP$: Using dIP verifier V, construct NP verifier $\widetilde{V}$: On input x and certificate $a_1, a_2, a_3, \ldots, a_{k(n)}$, $\widetilde{V}$ checks that all odd numbered messages a_i are consistent with what V would output. In other words, $\widetilde{V}$ checks that $a_1 = V(x), a_3 = V(x, a_1, a_2), a_5 = V(x, a_1, a_2, a_3, a_4)$, and so on. If the odd numbered messages are consistent, then, $\widetilde{V}$ outputs $V(x, a_1, a_2, a_3, \ldots, a_{k(n)})$; If the odd numbered messages are not consistent, then $\widetilde{V}$ outputs 0. We can see that if $V(x) = 1$, then such a certificate consisting of the the messages would exist, and $\widetilde{V}(x) = 1$.

So if deterministic interactive proofs don't work, what will?

2.3 (Probabilistic) Interactive Proofs

We modify the deterministic interactive proof system defined above:

L is in $IP[k(n)]$ if $\exists V$ such that:

$$x \in L \Rightarrow \exists P \; s.t. \; Pr[out_V \langle V, P\rangle(x) = 1] > \frac{2}{3} \qquad \text{(Completeness)}$$
$$x \notin L \Rightarrow \forall P, \; Pr[out_V \langle V, P\rangle(x) = 1] < \frac{1}{3} \qquad \text{(Soundness)}$$

Note that the randomness lies with the verifier and not the prover because there is always a best strategy for the prover to take to trick the verifier.

We note that the following is a landmark theorem in complexity theory.

Theorem 2.4. *IP = PSPACE.*

Subject to time constraints, we may see some part of the proof of the above theorem in a future lecture. For now, let us investigate the power of IP.

Definition 2.5. $GNI = \{\langle G_0, G_1\rangle : G_0 \not\approx G_1\}$*, where* $G_0 \approx G_1$ *if* $\exists \pi : V \to V$ *such that* $\pi(G_0) = G_1$.

Claim 2.6. *Graph Non-Isomorphism (GNI)* $\in IP$

Algorithm:

On input of G_0, G_1, V flips a bit b, picks a random permutation π, and sends $\pi(G_b)$ to P. P then sends back b', representing whether it thinks that $G_b \approx G_0$ or $G_b \approx G_1$. Then, V accepts if $b = b'$.

To prove that this algorithm works, we consider its completeness and soundness. Completeness: First, suppose $G_0 \not\approx G_1$. Then, the prover can distinguish whether G_b is a permutation of G_0 or a permutation of G_1 and send the correct b' back. In this case, $b' = b$ with probability 1 and V will always accept. Soundness: Now, suppose $G_0 \approx G_1$. In this case, the prover cannot distinguish whether G_b is a permutation of G_0 or G_1 since it is isomorphic to both, and can only guess. Thus, it sends back 0 or 1 with probability 0.5 each, so $b' = b$ with probability 0.5. Although the 0.5 acceptance probability for isomorphic graphs falls below the $\frac{1}{3}$ bound, the algorithm can be repeated over and over again to drive the probability down. Thus, we have shown that $GNI \in IP$

Note that we have assumed that V's coins are unknown to P. This is called a private randomness setting. However, as we will see in the next lecture, this relaxing this assumption to a public randomness setting does not affect the proof.