# 1 Introduction

In Lecture 26, we proved that each term in the simply typed $\lambda$-calculus would never get stuck. Today, we want to show that it will actually terminate. This property is known as *strong normalization*.

Formally, we want to prove that if $\vdash e : \tau$, then $e \Downarrow$. We will prove this by induction, but we will need a fairly sophisticated induction hypothesis that takes both the typing and the reduction order into account. We cannot just do induction on the subterm relation. For example, even if $e_1$ and $e_2$ terminate, we cannot conclude that $e_1\, e_2$ does: consider $e_1 = e_2 = \lambda x.\, xx$.

# 2 Church vs. Curry

We will prove this theorem in the pure simply-typed $\lambda$-calculus in Curry style. This differs from Church style in that the binding occurrence of a variable in a $\lambda$-abstraction is not annotated with its type.

Let $\alpha, \beta, \ldots$ denote type variables, $x, y, \ldots$ term variables, $\sigma, \tau, \ldots$ types, and $d, e, \ldots$ terms. In the Curry-style simply typed $\lambda$-calculus, terms and types are defined by

$$e \;::=\; x \;\mid\; e_1\, e_2 \;\mid\; \lambda x.\, e \qquad\qquad \tau \;::=\; \alpha \;\mid\; \sigma \to \tau$$

and the typing rules are

$$\Gamma,\, x:\tau \vdash x:\tau \qquad \frac{\Gamma \vdash e:\sigma \to \tau \quad \Gamma \vdash d:\sigma}{\Gamma \vdash (e\, d):\tau} \qquad \frac{\Gamma,\, x:\sigma \vdash e:\tau}{\Gamma \vdash (\lambda x.\, e):\sigma \to \tau}$$

Note that in Church style, a closed term can have at most one type, but in Curry style, if it has any type at all, then it has infinitely many. For example, $\vdash \lambda x.\, x : ((\alpha \to \beta) \to \gamma) \to ((\alpha \to \beta) \to \gamma)$. In general, if $\vdash e:\tau$, then also $\vdash e:\tau'$, where $\tau'$ is any substitution instance of $\tau$.

A term $e$ is *typable* if there exists a type environment $\Gamma$ and a type $\tau$ such that $\Gamma \vdash e:\tau$. One can show by induction that if $\Gamma \vdash e:\tau$, then $FV(e) \subseteq \mathsf{dom}\,\Gamma$.

# 3 Strong Normalization

By the Church–Rosser theorem, normal forms are unique up to $\alpha$-equivalence, so any two reduction strategies starting from the same term that terminate must yield the same result up to $\alpha$-equivalence. However, there may be some strategies that terminate and some that do not.

A term is *strongly normalizing* (SN) if all $\beta$-reduction sequences starting from that term converge to a normal form; equivalently, if there is no infinite $\beta$-reduction sequence starting from that term. Our main theorem is

**Theorem 1.** *All typable terms are strongly normalizing.*

## 3.1 Ultra-Strong Normalization

We say that a term $e$ is *ultra-strongly normalizing with respect to* $\Gamma$ *and* $\sigma$ and write $\Gamma \vdash_{USN} e : \sigma$ if

(i) $\Gamma \vdash e : \sigma$

(ii) for all $n \geq 0$, if $\sigma$ is of the form $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n \to \tau$ and $\Gamma \vdash_{\scriptscriptstyle USN} e_i : \sigma_i$, $1 \leq i \leq n$, then $e\ e_1\ e_2\ \cdots\ e_n$ is SN.

A term $e$ is *ultra-strongly normalizing* (USN) if it is ultra-strongly normalizing with respect to some $\Gamma$ and $\sigma$.

The definition of the relation $\vdash_{\scriptscriptstyle USN}$ may seem circular, but it is not: $\Gamma \vdash_{\scriptscriptstyle USN} e : \sigma$ is defined in terms of $\Gamma \vdash_{\scriptscriptstyle USN} e_i : \sigma_i$, where the $\sigma_i$ are strict subexpressions of $\sigma$, so it is well-defined by structural induction on types.

Almost all the work we need to do is contained in the following lemma:

**Lemma 2.** *Let $x_1, \ldots, x_n$ be distinct variables. If*

(i) $\Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash e : \tau$,

(ii) $\Gamma \vdash_{\scriptscriptstyle USN} d_i : \sigma_i$, $1 \leq i \leq n$, *and*

(iii) $x_j \notin FV(d_i)$ *for* $j > i$,

*then* $\Gamma \vdash_{\scriptscriptstyle USN} e\{d_1/x_1\} \cdots \{d_n/x_n\} : \tau$.

*Proof.* Suppose the three premises (i)–(iii) hold. The proof is by induction on the structure of $e$.

**Case 1** Variable $x$.

**Case 1A** $x = x_i$ for some $i$. We have $\tau = \sigma_i$ by assumption (i) and $x\{d_1/x_1\} \cdots \{d_n/x_n\} = d_i$ by assumption (iii). The desired conclusion is therefore $\Gamma \vdash_{\scriptscriptstyle USN} d_i : \sigma_i$, which follows from assumption (ii).

**Case 1B** $x \notin \{x_1, \ldots, x_n\}$. We have $\Gamma \vdash x : \tau$ by assumption (i), and $x\{d_1/x_1\} \cdots \{d_n/x_n\} = x$. The desired conclusion is therefore $\Gamma \vdash_{\scriptscriptstyle USN} x : \tau$. We already have $\Gamma \vdash x : \tau$, so we need only show that $x\ e_1\ \cdots\ e_m$ is SN for all appropriately typed USN terms $e_i$. But in any infinite $\beta$-reduction sequence starting from $x\ e_1\ \cdots\ e_m$, every reduction must be inside one of the $e_i$, since there are no other $\beta$-redexes; therefore some $e_i$ must contain an infinite subsequence. But this is impossible, since the $e_i$ are USN.

**Case 2** Application $e_1\ e_2$. For some type $\sigma$,

$$\begin{aligned}
&\Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash (e_1\ e_2) : \tau \\
&\quad\Rightarrow\quad \Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash e_1 : \sigma \to \tau \ \wedge\ \Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash e_2 : \sigma \\
&\quad\Rightarrow\quad \Gamma \vdash_{\scriptscriptstyle USN} e_1\{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \to \tau \ \wedge\ \Gamma \vdash_{\scriptscriptstyle USN} e_2\{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma
\end{aligned} \tag{1}$$

by the induction hypthesis. By clause (i) in the definition of USN, this implies

$$\begin{aligned}
&\Gamma \vdash e_1\{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \to \tau \ \wedge\ \Gamma \vdash e_2\{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \\
&\quad\Rightarrow\quad \Gamma \vdash (e_1\ e_2)\{d_1/x_1\} \cdots \{d_n/x_n\} : \tau \qquad .
\end{aligned}$$

This establishes clause (i) in the definition of USN for $e_1\ e_2$. For clause (ii), we must show that if $\tau = \tau_3 \to \cdots \to \tau_m$ and if $\Gamma \vdash_{USN} e_i : \tau_i$ for $3 \le i \le m$, then

$$\begin{aligned} &(e_1\ e_2)\{d_1/x_1\}\cdots\{d_n/x_n\}\ e_3\ \cdots\ e_m \\ &\quad = \quad (e_1\{d_1/x_1\}\cdots\{d_n/x_n\})\ (e_2\{d_1/x_1\}\cdots\{d_n/x_n\})\ e_3\ \cdots\ e_m \end{aligned} \tag{2}$$

is SN. But by (1),

$$\begin{aligned} &\Gamma \vdash_{USN} e_1\{d_1/x_1\}\cdots\{d_n/x_n\} : \sigma \to \tau_3 \to \cdots \to \tau_m \\ &\Gamma \vdash_{USN} e_2\{d_1/x_1\}\cdots\{d_n/x_n\} : \sigma \\ &\Gamma \vdash_{USN} e_i : \tau_i, \quad 3 \le i \le m, \end{aligned}$$

thus (2) is SN. This proves that $\Gamma \vdash_{USN} (e_1\ e_2)\{d_1/x_1\}\cdots\{d_n/x_n\} : \tau$.

**Case 3** Abstraction $\lambda x.\, e$. We can assume without loss of generality that $\lambda x.\, e$ has been $\alpha$-converted so that $x \notin FV(d_i)$ and $x \ne x_i$ for any $i$, $1 \le i \le n$. Instead of $x$, let us call this bound variable $x_{n+1}$. Then for some $\sigma_{n+1}$, we have

(i) $\Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash (\lambda x_{n+1}.\, e) : \sigma_{n+1} \to \tau$,

(ii) $\Gamma \vdash_{USN} d_i : \sigma_i$, $1 \le i \le n$, and

(iii) $x_j \notin FV(d_i)$ for $j > i$ (including $j = n+1$),

and we wish to show $\Gamma \vdash_{USN} (\lambda x_{n+1}.\, e)\{d_1/x_1\}\cdots\{d_n/x_n\} : \sigma_{n+1} \to \tau$.

Starting from assumption (i), we have

$$\begin{aligned} &\Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash (\lambda x_{n+1}.\, e) : \sigma_{n+1} \to \tau \\ &\quad \Rightarrow \quad \Gamma,\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1,\, x_{n+1} : \sigma_{n+1} \vdash e : \tau \\ &\quad \Rightarrow \quad \Gamma,\, x_{n+1} : \sigma_{n+1},\, x_n : \sigma_n,\, \ldots,\, x_1 : \sigma_1 \vdash e : \tau. \end{aligned}$$

If $d_{n+1}$ is any term such that $\Gamma \vdash_{USN} d_{n+1} : \sigma_{n+1}$, then by the induction hypothesis we have both

$$\Gamma,\, x_{n+1} : \sigma_{n+1} \vdash_{USN} e\{d_1/x_1\}\cdots\{d_n/x_n\} : \tau \tag{3}$$

$$\Gamma \vdash_{USN} e\{d_1/x_1\}\cdots\{d_{n+1}/x_{n+1}\} : \tau. \tag{4}$$

For clause (i) in the definition of USN, starting from (3), we have

$$\begin{aligned} &\Gamma,\, x_{n+1} : \sigma_{n+1} \vdash e\{d_1/x_1\}\cdots\{d_n/x_n\} : \tau \\ &\quad \Rightarrow \quad \Gamma \vdash \lambda x_{n+1}.\, (e\{d_1/x_1\}\cdots\{d_n/x_n\}) : \sigma_{n+1} \to \tau \\ &\quad \Rightarrow \quad \Gamma \vdash (\lambda x_{n+1}.\, e)\{d_1/x_1\}\cdots\{d_n/x_n\} : \sigma_{n+1} \to \tau \qquad \text{since } x_{n+1} \notin FV(d_i). \end{aligned}$$

For clause (ii), we wish to show that if in addition to the assumptions (i)–(iii) above, $\tau = \sigma_{n+2} \to \cdots \to \sigma_m \to \rho$ and $\Gamma \vdash_{USN} d_i : \sigma_i$, $n+1 \le i \le m$, then

$$\begin{aligned} &(\lambda x_{n+1}.\, e)\{d_1/x_1\}\cdots\{d_n/x_n\}\ d_{n+1}\ \cdots\ d_m \\ &\quad = \quad (\lambda x_{n+1}.\, (e\{d_1/x_1\}\cdots\{d_n/x_n\}))\ d_{n+1}\ \cdots\ d_m \end{aligned}$$

is SN. Consider any infinite reduction sequence starting from this term. We know that $e\{d_1/x_1\} \cdots \{d_n/x_n\}$ is SN by (3), and we know that the $d_i$ are SN by assumption, $n+1 \leq i \leq m$. Therefore, eventually a head reduction must be performed:

$$\begin{array}{ll} (\lambda x_{n+1}.\,(e\{d_1/x_1\} \cdots \{d_n/x_n\}))\; d_{n+1} \;\cdots\; d_m & \\ \quad \stackrel{*}{\rightarrow} \quad (\lambda x_{n+1}.\,(e\{d_1/x_1\} \cdots \{d_n/x_n\})')\; d'_{n+1} \;\cdots\; d'_m & \\ \quad \rightarrow \quad (e\{d_1/x_1\} \cdots \{d_n/x_n\})'\{d'_{n+1}/x_{n+1}\}\; d'_{n+2} \;\cdots\; d'_m. & \end{array}$$

But we could have done the head reduction initially:

$$\begin{array}{ll} (\lambda x_{n+1}.\,(e\{d_1/x_1\} \cdots \{d_n/x_n\}))\; d_{n+1} \;\cdots\; d_m & \\ \quad \rightarrow \quad e\{d_1/x_1\} \cdots \{d_n/x_n\}\{d_{n+1}/x_{n+1}\}\; d_{n+2} \;\cdots\; d_m & \\ \quad \stackrel{*}{\rightarrow} \quad (e\{d_1/x_1\} \cdots \{d_n/x_n\})'\{d'_{n+1}/x_{n+1}\}\; d'_{n+2} \;\cdots\; d'_m, & \end{array}$$

leading to an infinite reduction sequence from $e\{d_1/x_1\} \cdots \{d_n/x_n\}\{d_{n+1}/x_{n+1}\}\; d_{n+2} \;\cdots\; d_m$. But this contradicts (4). □

*Proof of Theorem 1.* Any typable term is USN: take $n = 0$ in Lemma 2. Any term that is USN is SN: take $n = 0$ in the definition of USN. □