Lecturer: Dexter Kozen

As observed in Lecture 19, one of the problems with modeling the untyped λ-calculus is that we cannot have a nontrivial domain D isomorphic to its function space $D \to D$ because of cardinality restrictions. However, Dana Scott showed that given any pointed CPO D, it is possible to embed D into a pointed CPO D_∞ that is isomorphic to its *continuous* function space $[D_\infty \to D_\infty]$. This construction allows us to give a denotational model of the untyped λ-calculus.

The notes on the following pages give an overview of Scott's construction. The notes are extracted from

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Leonid Rudin

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We say M = N iff it is provable from the axioms in (1.2).

1.2.1 Let $Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$. Then Y solves the problem: given any λ-term F, find a λ-term A such that A = FA: i.e., A is a fixed point of F.

Theorem

For any λ-term F there exists at least one fixed point A of F, namely

$$A = Y(F): \text{ i.e., } F(Y(F)) = Y(F).$$

This theorem is proved directly by applying Y to F and using the β-rule:

$$\begin{aligned} YF &= (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))(F) \\ &= (\lambda x.F(xx))(\lambda x.F(xx)) \\ &= F((\lambda x.F(xx))(\lambda x.F(xx))) \\ &= F(YF)\text{: i.e., } YF \text{ is a fixed point.} \end{aligned}$$

Y is called a paradoxical combinator.

B. Construction of D_∞.

Because of the type-free nature of the application of λ-calculus, the domain of any interpretation must include a significant portion of its own function space. Thus semantics of this calculus is based on a solution of an isomorphism

$$(*)\ D \simeq [D \to D],$$

where $[D \to D]$ denotes some suitable notion of function space from D to itself. Thus we need to construct a solution to the equation (*). Historically, a fist solution was found by D. Scott and was called D_∞. We now give some definitions and properties of D_∞. Again, for proofs the reader is referred to Barendregt (1977) and Wadsworth (1976).

1.3 Definition.

The partially ordered set $(D,\leq)$ is a complete lattice if $\forall X \subset D$, there exists a supremum $\bigcup X \in D$. Denote top $\top = \bigcup D$ and bottom $\perp = \bigcap D$ (i.e., largest and smallest elements of D). The supremum and infimum of $\{x,y\}$ are denoted as $x \cup y$, $x \cap y$.

1.4 Lemma

If D is a complete lattice, then each of its subsets X has an infimum: $\bigcap X = \bigcup\{z \mid z \leq X\}$. Here $z \leq X$ iff $\forall x \in X$, $z \leq x$.

1.5 Definition.

A subset $X \subset D$ is directed iff $\forall x,y \in X \; \exists \, z \in X$, such that $x,y \leq z$.

1.6 Definition

A and B are complete lattices. A mapping $f\colon A \to B$ is called continuous iff $f(\bigcup_A X) = \bigcup_B f(X)$ for all directed $X \subset A$.

Here $f(X) = \{y = f(x) \mid x \in X\}$. $\bigcup_D$ means that a supremum is taken in set D; we usually omit D, since it will be clear from context.

Let $[A \to B]$ be the set of all continuous functions from A to B.

1.7 Definition.

Let D and D' be complete lattices. Let $\Phi\colon D \to D'$, $\Psi\colon D' \to D$ (Φ,Ψ) is called a projection pair iff

(i) Φ,Ψ are continuous

(ii) $\forall x \in D$, $\Psi(\Phi(x)) = x$ or $\Psi \circ \Phi$ is an identity.

(iii) $\forall x \in D'$, $\Phi(\Psi(x)) \leq x$

1.8 Definition.

Let $(D,\sqsubseteq)$ be an arbitrary nontrivial complete lattice. Define $D_0 = D$, $D_{n+1} = [D_n \rightarrow D_n]$.

Define (Φ_n, Ψ_n) where $D_n \underset{\Psi_n}{\overset{\Phi_n}{\rightleftarrows}} D_{n+1}$, $n=0,1,2,\ldots$ where

$\Phi_0(x) = \lambda y \in D_0 . x, \quad x \in D_0$

$\Psi_0(x') = x'(\perp_{D_0})$; $x' \in D_1$; $\perp_{D_0}$ is $\sqcap D_0$.

And then, inductively, we define:

$\Phi_{n+1}(x) = \Phi_n \circ x \circ \Psi_n, \; x \in D_{n+1}$

$\Psi_{n+1}(x') = \Psi_n \circ x' \circ \Phi_n, \; x' \in D_{n+2}$

$$\begin{array}{ccccccccc}
\ldots \leftarrow & D_n & \xleftarrow{\Psi_n} & D_{n+1} & \xleftarrow{\Psi_{n+1}} & D_{n+2} & \longleftarrow & \ldots \\
 & \downarrow x & & \uparrow x' & & & & \\
\ldots \rightarrow & D_n & \xrightarrow[\Phi_n]{} & D_{n+1} & \xrightarrow[\Phi_{n+1}]{} & D_{n+2} & \longrightarrow & \ldots
\end{array}$$

The inverse limit of the D_n's is called D_∞; i.e.,

$$D_\infty = \{\langle x_n \rangle_{n=0}^{\infty} \mid x_n = \Psi_n(x_{n+1}),\ x_n \in D_n\}.$$

1.8.1 Lemma.

Let D be a complete lattice. Then $[D \rightarrow D]$ is a complete lattice under pointwise ordering: i.e., $f \sqsubseteq g \Leftrightarrow \forall x\, f(x) \sqsubseteq g(x)$.

Corollary.

$\forall_n$, D_n is a complete lattice.

Definition.

Let x and $y \in D_\infty$; $x = \langle x_n \rangle_{n=0}^{\infty}$ and $y = \langle y_n \rangle_{n=0}^{\infty}$. Then we say $x \sqsubseteq y$ iff $\forall n\ x_n \sqsubseteq y_n$.

1.8.2 Theorem

D_∞ is a complete lattice under the componentwise partial ordering $\sqsubseteq$.

1.9 Lemma

(i) $\forall n \geq 0$, (Φ_n, Ψ_n) is a projection pair

(ii) $\forall n \geq 0$, Φ_n and Ψ_n are distributive.

1.10 Definition

Define a set of mappings inductively:

$$\{\Phi_{n\infty}: D_n \to D_\infty \mid n=0,1,2,\ldots\}$$

and

$$\{\Psi_{\infty n}: D_\infty \to D_n \mid n=0,1,2,\ldots\}$$

(i) Define $\Phi_{0\infty}: D_0 \to D_\infty$

$\forall x_0 \in D_0 \quad \Phi_{0\infty}(x_0) = \langle y_n \rangle_{n=0}^{\infty} \in D_\infty$

such that $\quad y_0 = x_0,\ y_{n+1} = \Phi_n(y_n) \quad n=0,1,2,\ldots$

i.e., $\quad \Phi_{0\infty}(x_0) = (x_0, \Phi_0(x_0), \Phi_1 \circ \Phi_0(x_0), \ldots)$.

And define $\Psi_{\infty 0}: D_\infty \to D_0$ as follows:

$\forall x = \langle x_n \rangle_{n=0}^{\infty} \in D_\infty$, $\Psi_{\infty 0}(x) = x_0$.

(ii) Let $\Phi_{n+1\infty}: D_{n+1} \to D_\infty$.

Then define $\Phi_{n+1\infty}(x) = \Phi_{n\infty} \circ x \circ \Psi_{\infty n}$, $x \in D_{n+1}$

and $\Psi_{\infty n+1}: D_\infty \to D_{n+1}$ as

$\Psi_{\infty n+1}(x') = \Psi_{\infty n} \circ (x')^{\star} \circ \Phi_{n\infty}$, $x' \in D_{\infty}$

In the last equation $(x')^{\star}$ is used as a function on D_{∞}. It is defined in (1.11).

1.11 Definition

Let $x \in D_{\infty}$, $y \in D_{\infty}$: i.e., $x = \langle x_n \rangle_{n=0}^{\infty}$ and $y = \langle y_n \rangle_{n=0}^{\infty}$. Then $x^{\star}$ is a function from $D_{\infty} \rightarrow D_{\infty}$, such that for any y, $x^{\star}(y) = z \in D_{\infty}$, where $z = \langle z_n \rangle_{n=0}^{\infty}$ and

(i) $z_0 = \bigcup\{x_1(y_0), \Psi_0(x_2(y_1)), \ldots, \Psi_0 \circ \Psi_1 \circ \ldots \Psi_n(x_{n+2}(y_{n+1})) \ldots\}$

(ii) $z_n = \bigcup\{x_{n+1}(y_n), \Psi_n(x_{n+2}(y_{n+1})), \ldots \Psi_0 \circ \Psi_1 \circ \ldots \circ \Psi_{n+k}(x_{n+k+2}(y_{n+k+1})), \ldots\}$.

1.12 Lemma

Let $x \in D_{\infty}$. Then $x^{\star} \in [D_{\infty} \rightarrow D_{\infty}]$; i.e., $x^{\star}$ is a continuous function from D_{∞} to D_{∞}.

1.13 Lemma

$\forall f \in [D_{\infty} \rightarrow D_{\infty}]\ \exists\ x \in D_{\infty}$, such that $f = x^{\star}$.

So we see that (1.11) and (1.12) provide the following embedding: $x \rightarrow x^{\star}$ is the mapping $D_{\infty} \rightarrow [D_{\infty} \rightarrow D_{\infty}]$. We call it Φ. And Lemma (1.13) gives the mapping from $[D_{\infty} \rightarrow D_{\infty}]$ to D_{∞}. Call it Ψ.

1.14 Lemma

D_{∞} is homeomorphic to $[D_{\infty} \rightarrow D_{\infty}]$ under a pair of isomorphisms (Φ, Ψ), i.e., $D_{\infty} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} [D_{\infty} \rightarrow D_{\infty}]$.

1.15 Definition

$D_n^\infty = \{\Phi_{n\infty}(x) \mid x \in D_n\} \sqsubseteq D_\infty$.

D_n^∞ is a subspace of D_∞, and obviously $D_n^\infty \simeq D_n$.

Let $P_n \equiv \Phi_{n\infty} \circ \Psi_{\infty n} : D_\infty \rightarrow D_n^\infty$: i.e., a projection from the space D_∞ into the subspace D_n^∞. For convenience, we write x_n for $P_n(x)$, $x \in D_\infty$, and we write $x(y)$ for $x^{\star}(y)$ and f for $\Psi(f)$.

1.16 Lemma

For all x and y $\in D_\infty$:

(i) $\bot(y) = \bot = \sqcap D_\infty$

(ii) $\top(y) = \top = \sqcup D_\infty$

(iii) $x = y$ iff $x_n = y_n$ for all $n \geq 0$.

1.17 Lemma

$D_0^\infty \sqsubseteq D_1^\infty \sqsubseteq D_2^\infty \sqsubseteq \ldots \sqsubseteq D_n^\infty \sqsubseteq \ldots \sqsubseteq D_\infty$.

C. D_∞ as a model for λ-calculus

Since $D_\infty \simeq [D_\infty \rightarrow D_\infty]$, it is an appropriate model for a language where application is allowed without type limitations. Of course, then D_∞ is a possible model for λ-calculus, since if A and B are any terms, then A(B) and B(A) are perfectly defined terms. Thus, we should be able to map our language into the D_∞ model.

1.18 Definition

(i) Let ρ: {variables} $\rightarrow D_\infty$, ρ is called a valuation in D_∞.

(ii) For $d \in D_\infty$ and x a variable, $\rho(d/x) = \rho'$, where ρ' is a valuation in D_∞, such that $\rho'(y) = \rho(y)$ if $y \not\equiv x$ and $\rho'(x) = d$.

(iii) The interpretation ℓ of any term A in D_∞ under ρ is denoted $\ell[\![A]\!](\rho)$ and is a mapping from the set of λ-terms into the D_∞. It is defined inductively:

(S1) $\ell[\![x]\!](\rho) = \rho(x)$, where x is a variable.

(S2) $\ell[\![MN]\!](\rho) = \Phi(\ell[\![M]\!](\rho))(\ell[\![N]\!](\rho))$.

(S3) $\ell[\![\lambda x.M]\!](\rho) = \Psi(\lambda d \in D_\infty . \ell[\![M]\!](\rho(d/x)))$.

1.19 Theorem

Axioms (ρ), (δ), (τ), (subst.), (α), (β), (η) of (1.2) are valid formulas under the D_∞ interpretation.

1.20 Theorem

D_∞ is a model for λ-calculus.

D. A characterization of $\sqsubseteq$ in D_∞.

The following is due to Wadsworth (1976).

1.21 Definition

Each λ-term M is of the form $M \equiv \lambda x_1 \ldots x_n . (\lambda x.P) QM_1 \ldots M_m$ or $M \equiv \lambda x_1 \ldots x_n . x_i A_1 A_2 \ldots A_k$. $(\lambda x.P)$ is called the head redex of M. In the second case x_i is the head variable of M and M is said to be in a head normal form (compare with the definition of the normal form in (1.1)).

1.22 Definition

Let M and N be terms. Then we say $M \sqsubseteq N$ iff $\ell[\![M]\!](\rho) \sqsubseteq \ell[\![N]\!](\rho)$ for any valuation $\rho \in (Var \to D_\infty)$. Var is a set of all variables.

Lemma

$$M \sqsubseteq N \iff C[M] \sqsubseteq C[N] \text{ for all contexts } C[\ ].$$

1.2 3 Definition

We say that $M \leq_W N$ iff, for all contexts C[ ]:

if C[M] has a head normal form,
then C[N] has the same head normal form.

Here $\leq_W$ stands for Wadsworth's ordering relation.

1.2 4 Theorem (Wadsworth)

For all terms M and N

$$M \leq_W N \iff M \sqsubseteq N.$$

Here we would like to emphasize the importance of this theorem. Theorem (1.2 4) establishes the relation between the purely syntactical concept of the "head normal form" and the semantical relation "$\sqsubseteq$" in D_∞. Thus, we have this important link between syntactical form and semantical content. Historically, this kind of property made mathematics a deductive science and created the science of mathematical logic. With this in mind, we attempt to explore this link and the deductive reasoning system behind it.