

1 Outline

- Continuation-passing style (CPS)
- CPS semantics
- CPS conversion
- uML error checking

2 Continuation-Passing Style

Consider the statement `if $x \leq 0$ then x else $x+1$` . We can think of this as $(\lambda y. \text{if } y \text{ then } x \text{ else } x+1) (x \leq 0)$. To evaluate this, we would first evaluate the argument $x \leq 0$ to obtain a Boolean value, then apply the function $\lambda y. \text{if } y \text{ then } x \text{ else } x+1$ to this value. The function $\lambda y. \text{if } y \text{ then } x \text{ else } x+1$ is called a *continuation*, because it specifies what is to be done with the result of the current computation in order to continue the computation.

Given an expression e , it is possible to transform the expression into a function that takes a continuation k and applies it to the value of e . The transformation is applied recursively. This is called *continuation-passing style* (CPS). There are a number of advantages to this style:

- The resulting expressions have a much simpler evaluation semantics, since the sequence of reductions to be performed is specified by a series of continuations. The next reduction to be performed is always uniquely determined, and the remainder of the computation is handled by a continuation. Thus evaluation contexts are not necessary to specify the evaluation order.
- In practice, function calls and function returns can be handled in a uniform way. Instead of returning, the called function simply calls the continuation.
- In a recursive function, any computation to be performed on the value returned by a recursive call can be bundled into the continuation. Thus every recursive call becomes tail-recursive. For example, the factorial function

$$\text{fact } n = \text{if } n = 0 \text{ then } 1 \text{ else } n * \text{fact } (n - 1)$$

becomes

$$\text{fact}' n k = \text{if } n = 0 \text{ then } k 1 \text{ else } \text{fact}' (n - 1) (\lambda v. k (n * v)).$$

One can show inductively that $\text{fact}' n k = k (\text{fact } n)$, therefore $\text{fact}' n \lambda x. x = \text{fact } n$. This transformation effectively trades stack space for heap space in the implementation.

- Continuation-passing gives a convenient mechanism for non-local flow of control, such as goto statements and exception handling.

3 CPS Semantics

Our grammar for the λ -calculus was:

$$e ::= x \mid \lambda x. e \mid e_0 e_1$$

Our grammar for the CPS λ -calculus will be:

$$v ::= x \mid \lambda x. e \qquad e ::= v_0 v_1 \cdots v_n$$

This is a highly constrained syntax. Barring reductions inside the scope of a λ -abstraction operator, the expressions v are all irreducible. The only reducible expression is $v_0 v_1 \cdots v_n$. If $n \geq 1$, there exactly one redex $v_0 v_1$, and both the function and the argument are already fully reduced. The small step semantics has a single rule

$$(\lambda x. e) v \rightarrow e\{v/x\},$$

and we do not need any evaluation contexts.

The big step semantics is also quite simple, with only a single rule:

$$\frac{e\{v/x\} \Downarrow v'}{(\lambda x. e) v \Downarrow v'}.$$

The resulting proof tree will not be very tree-like. The rule has one premise, so a proof will be a stack of inferences, each one corresponding to a step in the small-step semantics. This allows for a much simpler interpreter that can work in a straight line rather than having to make multiple recursive calls.

4 CPS Conversion

We now give a translation into the CPS calculus. The translation takes an arbitrary λ -term e and produces a CPS term $\llbracket e \rrbracket$ which is a function taking a continuation as argument. Intuitively, $\llbracket e \rrbracket k$ applies k to the value of e .

We want our translation to satisfy $e \xrightarrow{CBV}^* v \Leftrightarrow \llbracket e \rrbracket k \xrightarrow{CPS}^* \llbracket v \rrbracket k$ for primitive values v and any variable $k \notin FV(e)$, and $e \uparrow_{CBV} \Leftrightarrow \llbracket e \rrbracket k \uparrow_{CPS}$.

The translation is (adding numbers as primitive values):

$$\begin{aligned} \llbracket n \rrbracket k &\triangleq k n \\ \llbracket x \rrbracket k &\triangleq k x \\ \llbracket \lambda x. e \rrbracket k &\triangleq k (\lambda x. \llbracket e \rrbracket) = k (\lambda x k'. \llbracket e \rrbracket k') \\ \llbracket e_0 e_1 \rrbracket k &\triangleq \llbracket e_0 \rrbracket (\lambda f. \llbracket e_1 \rrbracket (\lambda v. f v k)). \end{aligned}$$

(Here we have defined $\llbracket e \rrbracket$ in the style of ML; thus $\llbracket e \rrbracket k \triangleq e'$ really means $\llbracket e \rrbracket \triangleq \lambda k. e'$.)

4.1 An Example

In the CBV λ -calculus, we have

$$(\lambda x y. x) 1 \rightarrow \lambda y. 1$$

Let's evaluate the CPS-translation of the left-hand side using the CPS evaluation rules.

$$\begin{aligned} \llbracket (\lambda x y. x) 1 \rrbracket k &= \llbracket \lambda x. \lambda y. x \rrbracket (\lambda f. \llbracket 1 \rrbracket (\lambda v. f v k)) \\ &\rightarrow (\lambda f. \llbracket 1 \rrbracket (\lambda v. f v k)) (\lambda x. \llbracket \lambda y. x \rrbracket) \\ &\rightarrow \llbracket 1 \rrbracket (\lambda v. (\lambda x. \llbracket \lambda y. x \rrbracket) v k) \\ &\rightarrow (\lambda v. (\lambda x. \llbracket \lambda y. x \rrbracket) v k) 1 \\ &\rightarrow (\lambda x. \llbracket \lambda y. x \rrbracket) 1 k \\ &\rightarrow \llbracket \lambda y. 1 \rrbracket k. \end{aligned}$$

5 Error Checking

Now let us use CPS semantics to augment our previously defined uML language translation so that it supports error checking. This time our translated expressions will be functions of ρ and k denoting an environment and a continuation, respectively. The term $\mathcal{E}\llbracket e \rrbracket \rho k$ represents the result of evaluating e in the environment ρ and sending the resulting value to the continuation k .

Assume that we have an encoding of variable names that will allow us to represent an environment ρ as a list of (name,value) pairs, along with lookup and update functions $LOOKUP \rho \text{ "x"}$ and $UPDATE \rho \text{ "x" } v$.

In addition, we want to catch type errors that may occur during the evaluation. We use the following tags to keep track of types:

booleans	0	empty list	2	functions	4
integers	1	pairs	3	error	5

A *tagged value* is a value paired with its type tag; for example, $(0, \text{true})$. Using these tagged values, we can now define a translation that incorporates runtime type checking:

$$\begin{aligned}
\mathcal{E}\llbracket x \rrbracket \rho k &\triangleq k (LOOKUP \rho \text{ "x"}) \\
\mathcal{E}\llbracket b \rrbracket \rho k &\triangleq k (0, b) \\
\mathcal{E}\llbracket n \rrbracket \rho k &\triangleq k (1, n) \\
\mathcal{E}\llbracket \text{nil} \rrbracket \rho k &\triangleq k (2, 0) \\
\mathcal{E}\llbracket (e_0, e_1) \rrbracket \rho k &\triangleq \mathcal{E}\llbracket e_0 \rrbracket \rho (\lambda x_0. \mathcal{E}\llbracket e_1 \rrbracket \rho (\lambda x_1. k (3, (x_0, x_1)))) \\
\mathcal{E}\llbracket \text{let } x = e_1 \text{ in } e_2 \rrbracket \rho k &\triangleq \mathcal{E}\llbracket e_1 \rrbracket \rho (\lambda p. \mathcal{E}\llbracket e_2 \rrbracket (UPDATE \rho \text{ "x" } p) k) \\
\mathcal{E}\llbracket \lambda x. e \rrbracket \rho k &\triangleq k (4, \lambda x k. \mathcal{E}\llbracket e \rrbracket (UPDATE \rho \text{ "x" } x) k) \\
\mathcal{E}\llbracket \text{error} \rrbracket \rho k &\triangleq k (5, 0).
\end{aligned}$$

Now a functional application can check that it is actually applying a function:

$$\mathcal{E}\llbracket e_0 e_1 \rrbracket \rho k \triangleq \mathcal{E}\llbracket e_0 \rrbracket \rho (\lambda p. \text{let } (t, f) = p \text{ in if } t \neq 4 \text{ then error else } \mathcal{E}\llbracket e_1 \rrbracket \rho (\lambda v. f v k))$$

We can simplify this by defining a helper function `check-fn`:

$$\text{check-fn} \triangleq \lambda k p. \text{let } (t, f) = p \text{ in if } t \neq 4 \text{ then error else } k f.$$

The helper function takes in a continuation and a tagged value, checks the type, strips off the tag, and passes the raw (untagged) value to the continuation. Then

$$\mathcal{E}\llbracket e_0 e_1 \rrbracket \rho k \triangleq \mathcal{E}\llbracket e_0 \rrbracket \rho (\text{check-fn } (\lambda f. \mathcal{E}\llbracket e_1 \rrbracket \rho (\lambda v. f v k))).$$

Similarly,

$$\begin{aligned}
\mathcal{E}\llbracket \text{if } e_0 \text{ then } e_1 \text{ else } e_2 \rrbracket \rho k &\triangleq \mathcal{E}\llbracket e_0 \rrbracket \rho (\text{check-bool } (\lambda b. \text{if } b \text{ then } \mathcal{E}\llbracket e_1 \rrbracket \rho k \text{ else } \mathcal{E}\llbracket e_2 \rrbracket \rho k)) \\
\mathcal{E}\llbracket \#n e \rrbracket \rho k &\triangleq \mathcal{E}\llbracket e \rrbracket \rho (\text{check-pair } (\lambda t. k (\#n t))).
\end{aligned}$$