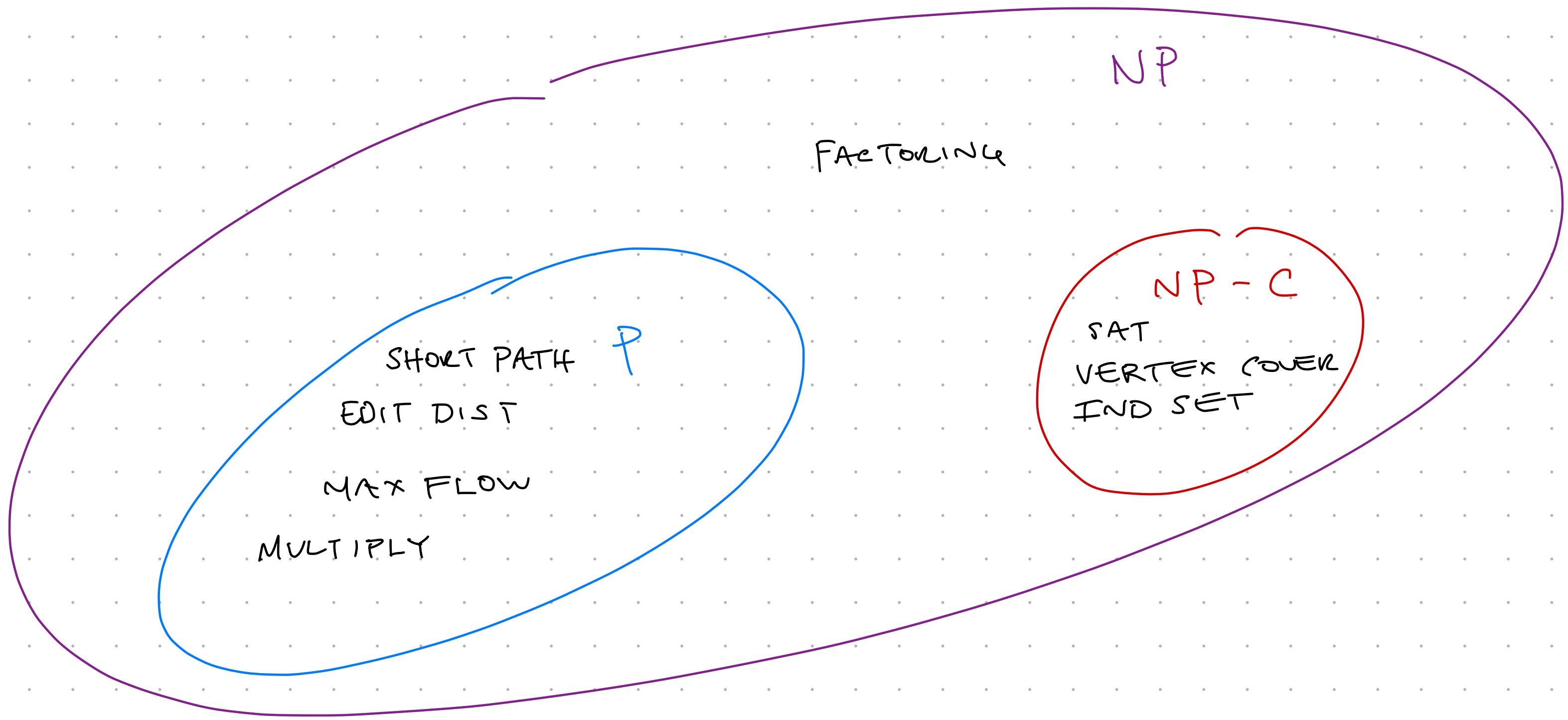
9 April 2025

Integer Linear Programming

Plan

* Approximating Minimum Vertex Cover.
* Announcements
* Integer Linear Programs



What do we do with NP-Hard problems?

1. Give up!
2. Solve SAT (per Lecture 28)
3. Solve Approximately

Minimum Weighted Vertex Cover

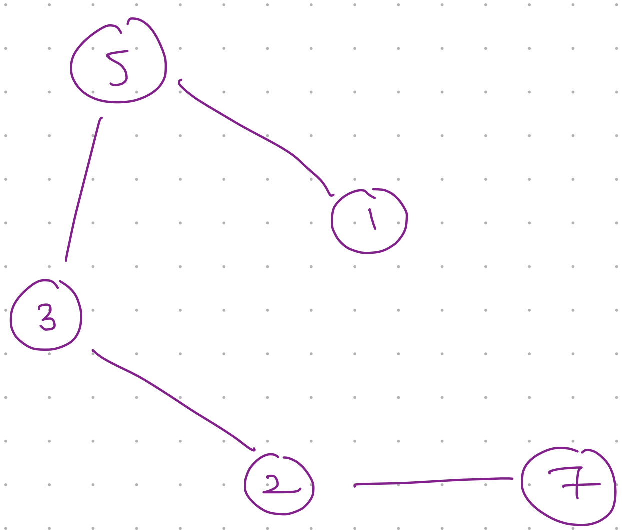
Given: Undirected Graph $G = (V, E)$

w/ vertex weights $W = \{w_v\}_{v \in V}$

$w_v \geq 0$

Find: minimum weight

Vertex Cover $C \subseteq V$ $\sum_{v \in C} w_v$.



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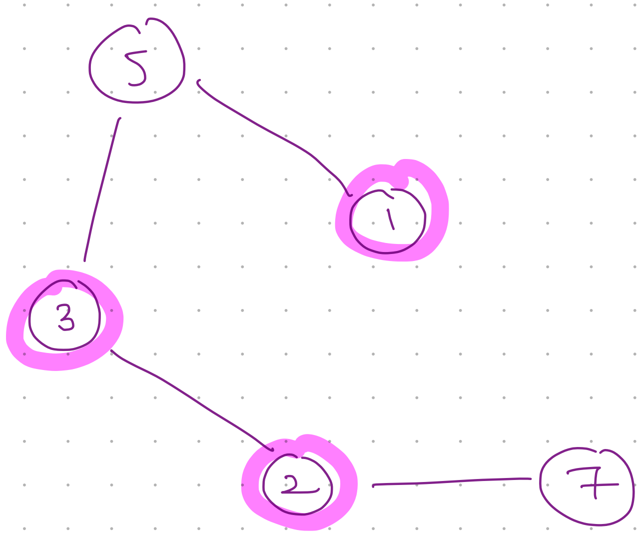

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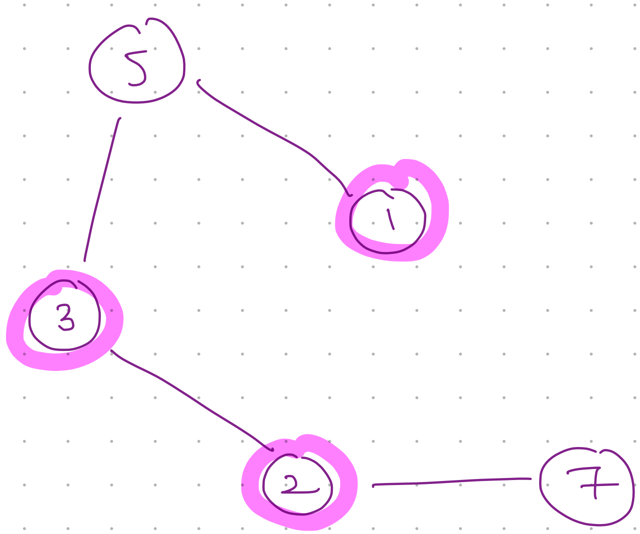

Thm Min Wt. VC is NP-Hard.

Pf. $\{w_v = 1 : v \in V\}$
recovers VERTEX COVER problem.

Approximate Minimum Weighted Vertex Cover

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Vertex Cover $C \subseteq V$

$$W(C) = \sum_{v \in C} w_v .$$

$$W^*(G) = \min_{C^* \subseteq V} W(C^*)$$

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Approximation Ratio of Algorithm A

(for minimization problem)

$$\boxed{r_A = \max_G \frac{W(A(G))}{W^*(G)}}$$

Today New paradigm for approximation algos.

* Integer Linear Programming. (ILP)

  ↳ Exact version : NP-Hard

  ↳ Relaxed version : Linear Programming $\in P$

  (generalizes MAXFLOW)

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* Integer Linear Programming. (ILP)

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(generalizes MAX FLOW)

* Paradigm:

— Write down ILP

— Solve corresponding LP

— Round solutions to be integral

Announcements

* Prelim 2 Graded $\longrightarrow$ Released After Lecture.

* HW 7 Released Today $\longrightarrow$ 2 problems.

Integer Linear Programming

variables $\quad x_1, x_2, \ldots, x_n$

linear inequalities

$$\langle a_i, x \rangle = \sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i \qquad i = 1 \ldots m$$

integer constraints

$$x_j \in \mathbb{Z}$$

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Linear optimization:

$$\text{Find } \min_{\vec{x}} \langle c, x \rangle = \sum_{j=1}^{n} c_j \cdot x_j$$

s.t. constraints.

Thm ILP is NP-Hard

Pf. By reduction from MIN WT VERTEX COVER.

Variables

Constraints

Objective

Thm ILP is NP-Hard

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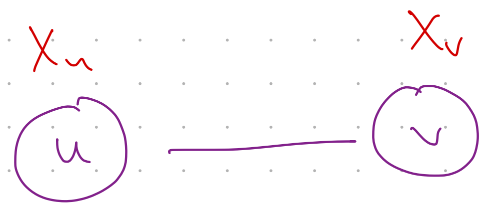
Variables per vertex $\{x_v : v \in V\}$

x_v indicates whether $v \in C$ or NOT.

Constraints



$x_v \in \{0,1\} \quad \forall v \in V.$

$x_u + x_v \geq 1 \quad \forall (u,v) \in E.$

Objective

$$\min_{\vec{x}} \sum_{j=1}^{n} w_j \cdot x_j$$

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Claim.

- $x \in \{0,1\}^n$ in 1-to-1 correspondence w/ $S \subseteq V$
- x is feasible for ILP $\iff$ S is a vertex cover.
- So ILP finds min wt. VC.

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~~Integer~~ Linear Programming

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~~integer constraints~~

$$\cancel{x_j \in \mathbb{Z}} \qquad x_j \in \mathbb{R}$$

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Linear Programming

variables $x_1, x_2, \ldots, x_n$ $x_j \in \mathbb{R}$

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Theorem Linear Programming $\in P$.

See CS 6820

Linear Programming is Useful!

Ex. Max Flow reduces to Linear Programming

Variables

Constraints

Objective

Linear Programming is Useful!

Ex. Max Flow reduces to Linear Programming

Variables

$$\{ f_e : e \in E \}$$

Constraints

$$0 \le f_e \le c_e \qquad \forall e \in E$$

$$\sum_{e=(u,v)} f_e = \sum_{e'=(v,w)} f_{e'} \qquad \forall v \in V \setminus \{s,t\}$$

Objective

$$\max \sum_{e=(s,v)} f_e$$

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Ex. Linear Systems reduces to Linear Programming

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Linear Programming is Useful!

Ex. Linear Systems reduces to Linear Programming

Variables $x_1 \, . \, . \, . \, x_n \in \mathbb{R}$

Constraints

$$\langle a_i, x \rangle \leq b_i$$

$$-\langle a_i, x \rangle \leq -b_i$$

Objective

$$\min_{\vec{x}} \quad 1$$

Linear Programming Relaxation.

Min Wt. Vertex Cover ILP

Variables $\{ x_v : v \in V \}$

Constraints

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Min Wt. Vertex Cover ~~I~~LP

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Can we give a guarantee on the quality of LP solution for solving ILP?

Approximate VC

On input $G = (V, E)$. Construct & solve LP

Variables $\{ x_v : v \in V \}$

Linear Inequalities

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Objective

$$\min_{\vec{x}} \sum_{j=1}^{n} w_j \cdot x_j$$

Then, ROUND solution

$C = \phi$

For $v \in V$.

if $x_v \ge 1/2$, $C \leftarrow C \cup \{v\}$

Return C

Theorem: Approximate VC returns a 2-approximate VC.

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Claim C is a vertex cover.

Claim $w(C) \leq 2 \cdot w^*(G)$

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* For all edges $(u,v) \in E$ $\quad x_u + x_v \geq 1$

$\Rightarrow$ at least one of $x_u, x_v \geq 1/2$

$\Rightarrow$ at least one of $u, v \in C$.

Claim $W(C) \leq 2 \cdot W^*(G)$

Theorem: Approximate VC returns a 2-approximate VC.

Claim C is a vertex cover.

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$\implies$ at least one of $x_u, x_v \geq 1/2$

$\implies$ at least one of $u, v \in C$.

Claim $W(C) \leq 2 \cdot W^*(G)$

$$W(C) = \sum w_j \cdot \mathbb{1}[x_j \geq 1/2]$$

$$\leq 2 \cdot \sum w_j \cdot x_j$$

$$= 2 \cdot OPT(VC\text{-}LP)$$

$$\leq 2 \cdot OPT(VC\text{-}ILP) = 2 \cdot [\text{Min Wt. VC}]$$