21 March 2025

Hamiltonian Path.

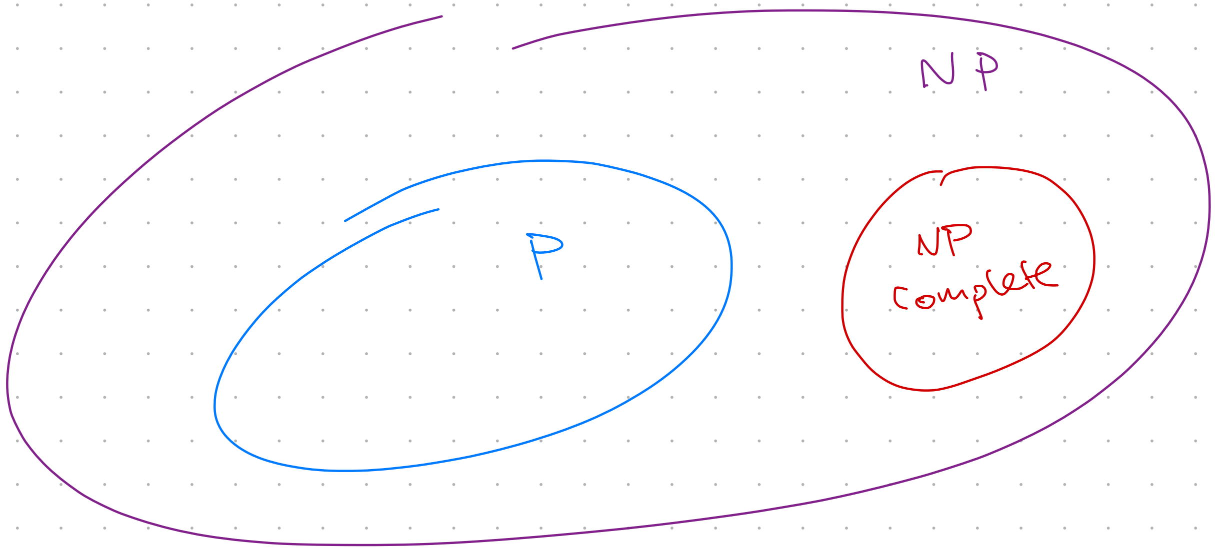
Plan

* Review of NP-Hard Problems
* Announcements
* Hamiltonian Path

# P vs NP, so far

We believe

$P \neq NP$




P = problems that can be solved in polynomial time

NP = problems verifiable in polynomial time

NP-Hard = Every problem in NP reduces in poly-time to these

## P vs NP, so far

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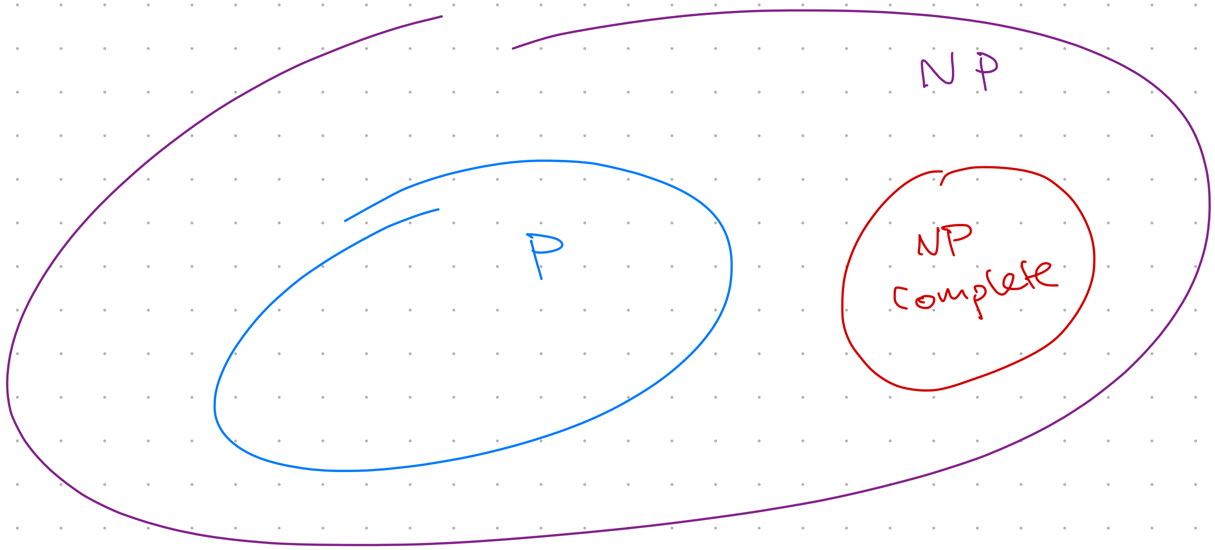


$\Rightarrow$ NP-Complete problems CANNOT be solved in poly-time

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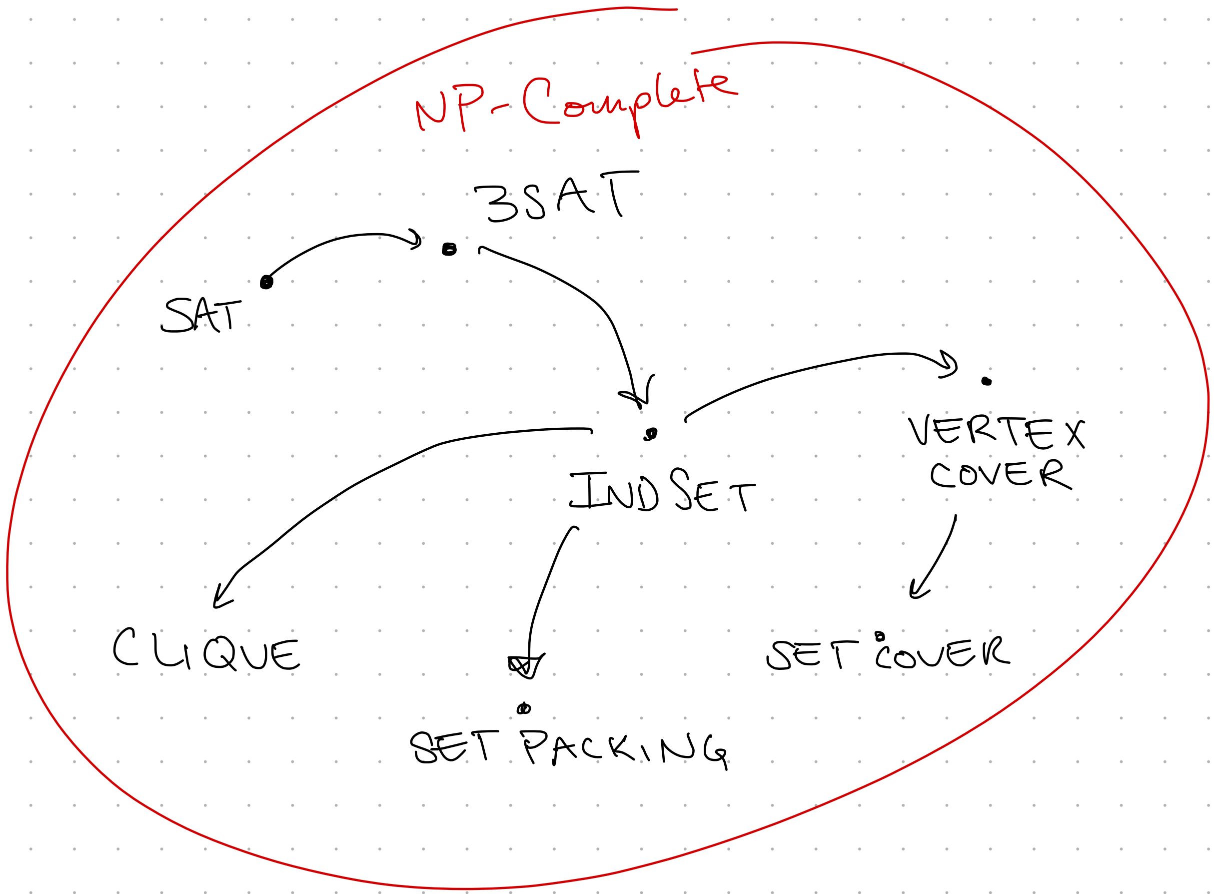
NP-Complete
3SAT
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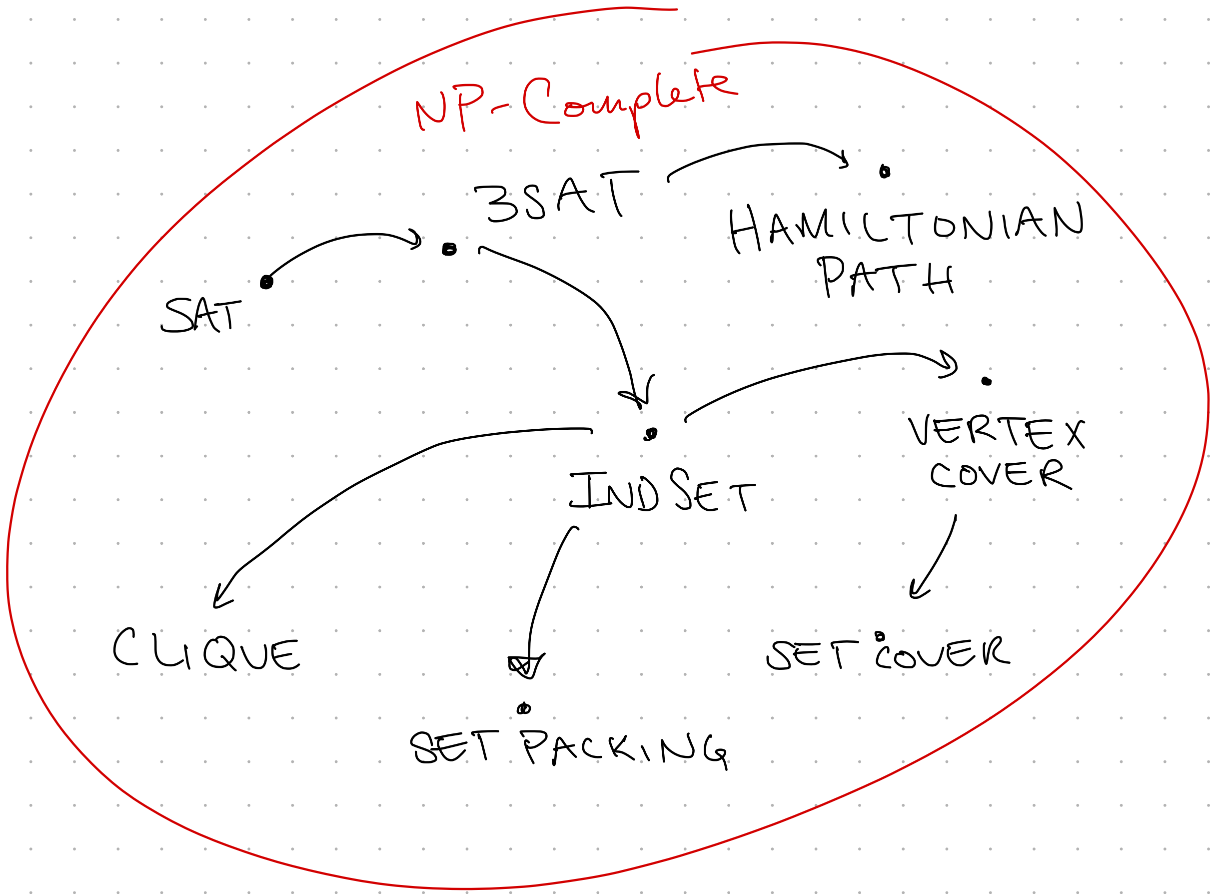

NP-Complete
3SAT
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HAMILTONIAN PATH
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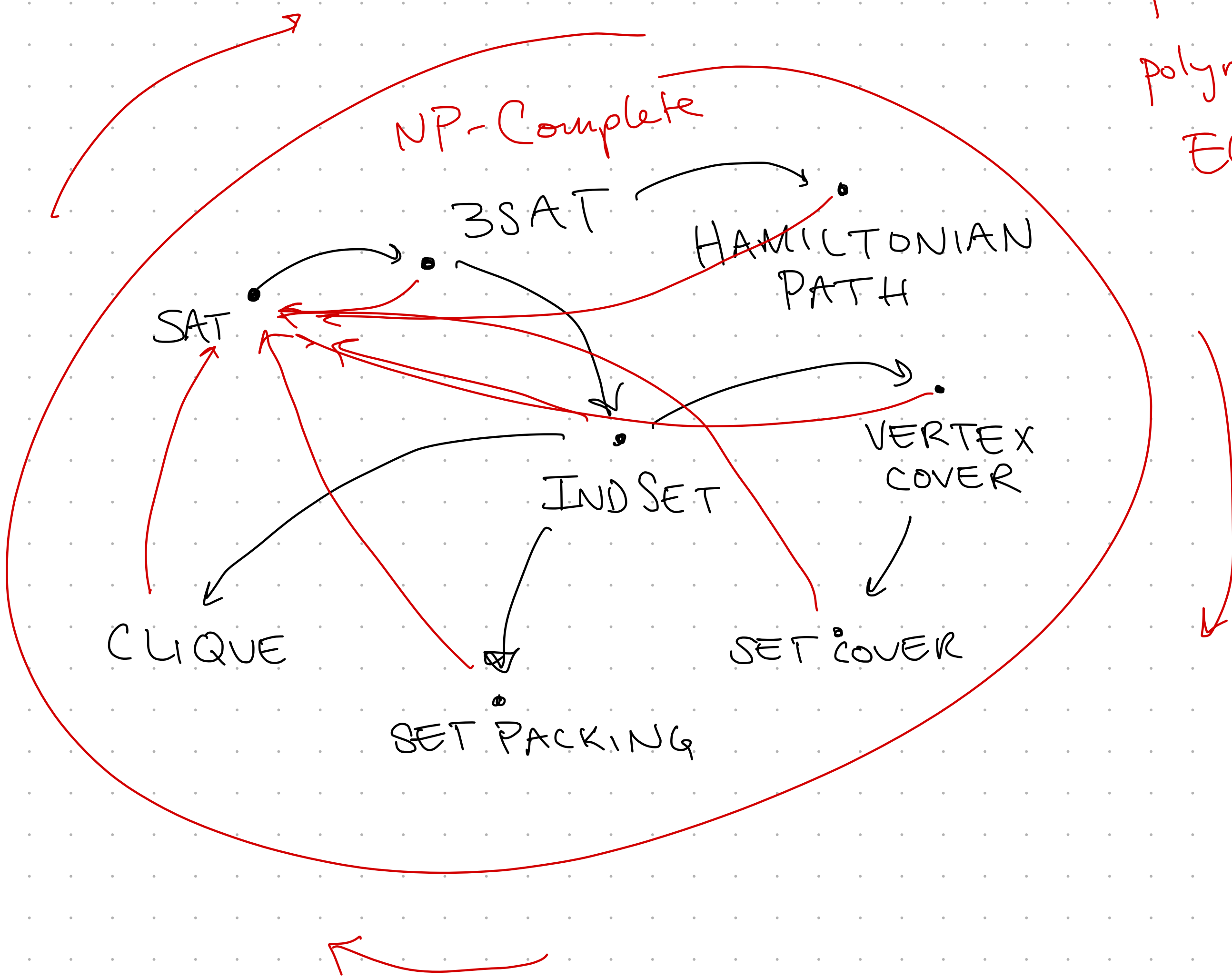



All NP-Complete problems are polynomial-time EQUIVALENT.

If any NP-Complete problem has poly-time algo $\Longrightarrow$ $P = NP$

If any NP-Complete problem can't be solved in poly-time $\Longrightarrow$ $P \neq NP$.

# Announcements

* HW6 out — only 2 problems

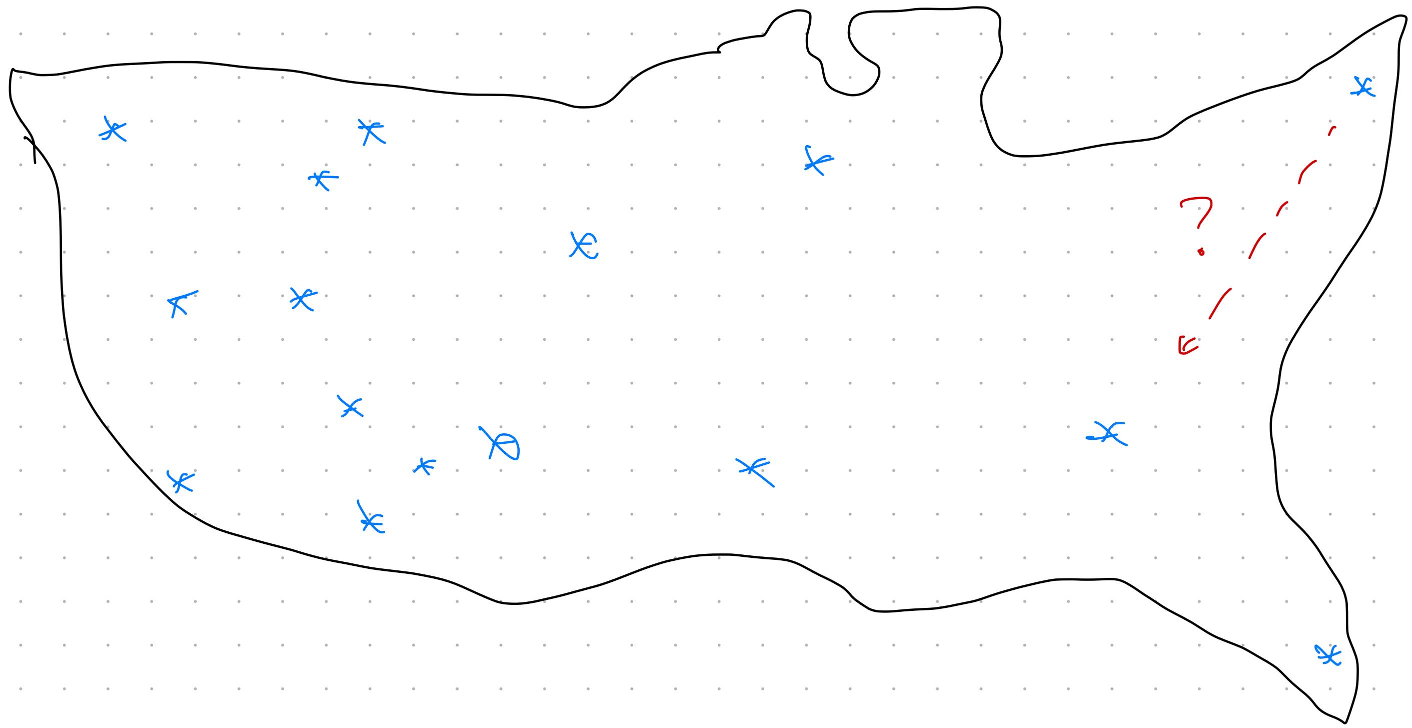
* Practice Exam problems released

* Review Session
  - ↳ Tues 7-9 p Gates G01.
  - ↳ No Saturday Recitation

* Prelim 2
  - ↳ Thurs 7:30 - 9 p.
  - ↳ Same location as last time.

Motivation.

* Planning a vacation
* Want to visit every National Park.
* Want to minimize driving time.



## Motivation.

* Planning a vacation
* Want to visit every National Park.
* Want to minimize driving time.

↳ Known as the Traveling Salesperson problem. (TSP)

Given a list of locations & driving time between each location,

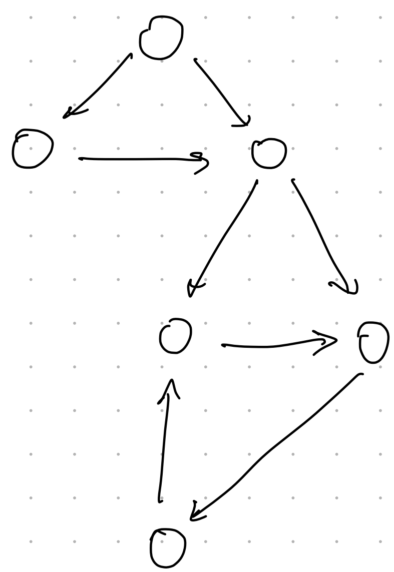
find shortest route that hits every location.

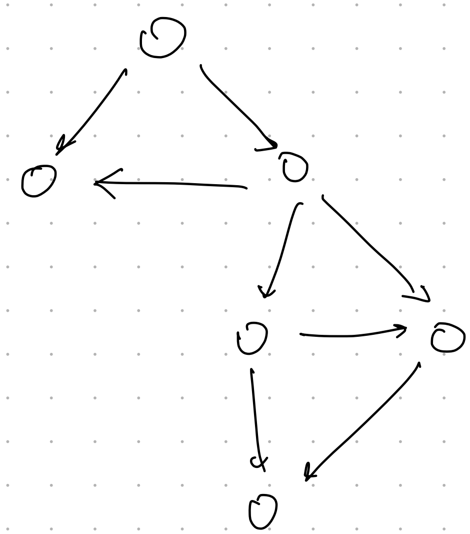
Today Closely Related Problem

## Hamiltonian Path

Given a directed graph $G = (V, E)$

Does there exist a simple path through $G$ that visits every vertex in $V$?





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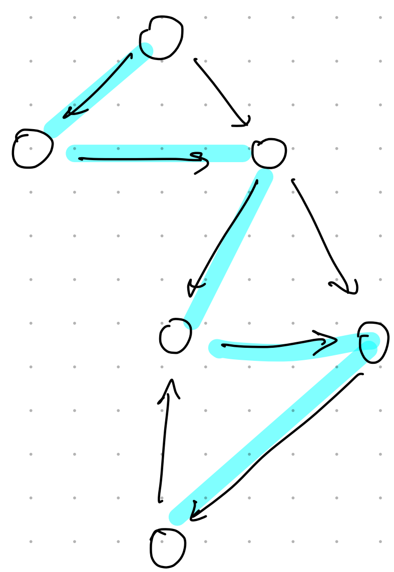
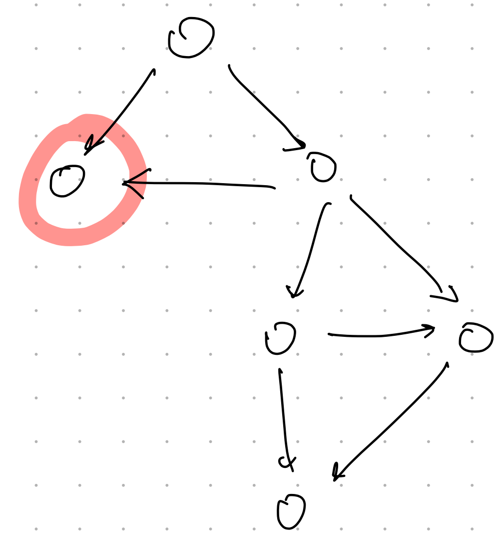


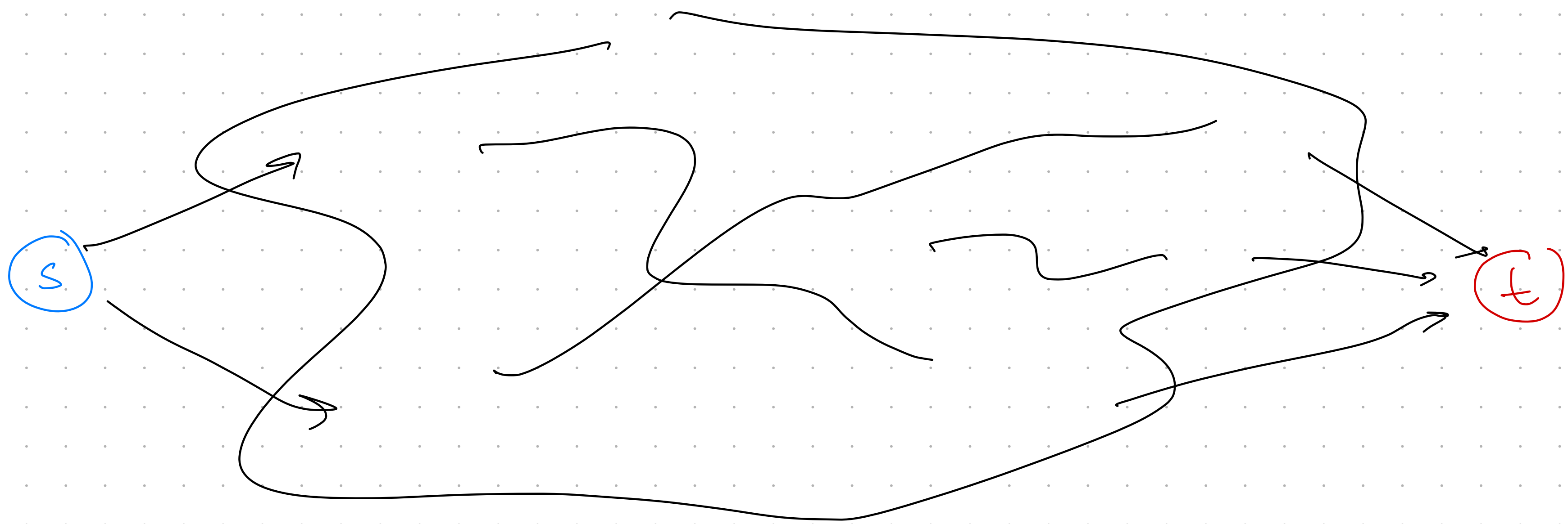


# Hamiltonian Path

Given a directed graph $G = (V, E)$

Does there exist a simple path through $G$ that visits every vertex in $V$?

WLOG Assume we're given a start vertex $s \in V$ and finish vertex $t \in V$.



Compare

* Shortest Path $\quad \exists$ an st-path of length $\leq k$?

EASY!

## Compare

* **Shortest Path** $\exists$ an $st$-path of length $\leq k$?

  EASY!

* **Longest Path** $\exists$ an $st$-path of length $\geq k$?

  HARD!

## Theorem

$$\text{SAT} \leq_p \text{HAM PATH} \leq_p \text{LONGEST PATH}$$

## Reduction from SAT

* Gadgets :
  - Assignments to variables
  - Ensure Clauses are Satisfied.

# Reduction from SAT

* Gadgets:
  - Assignments to variables
  - Ensure Clauses are Satisfied.

## Idea

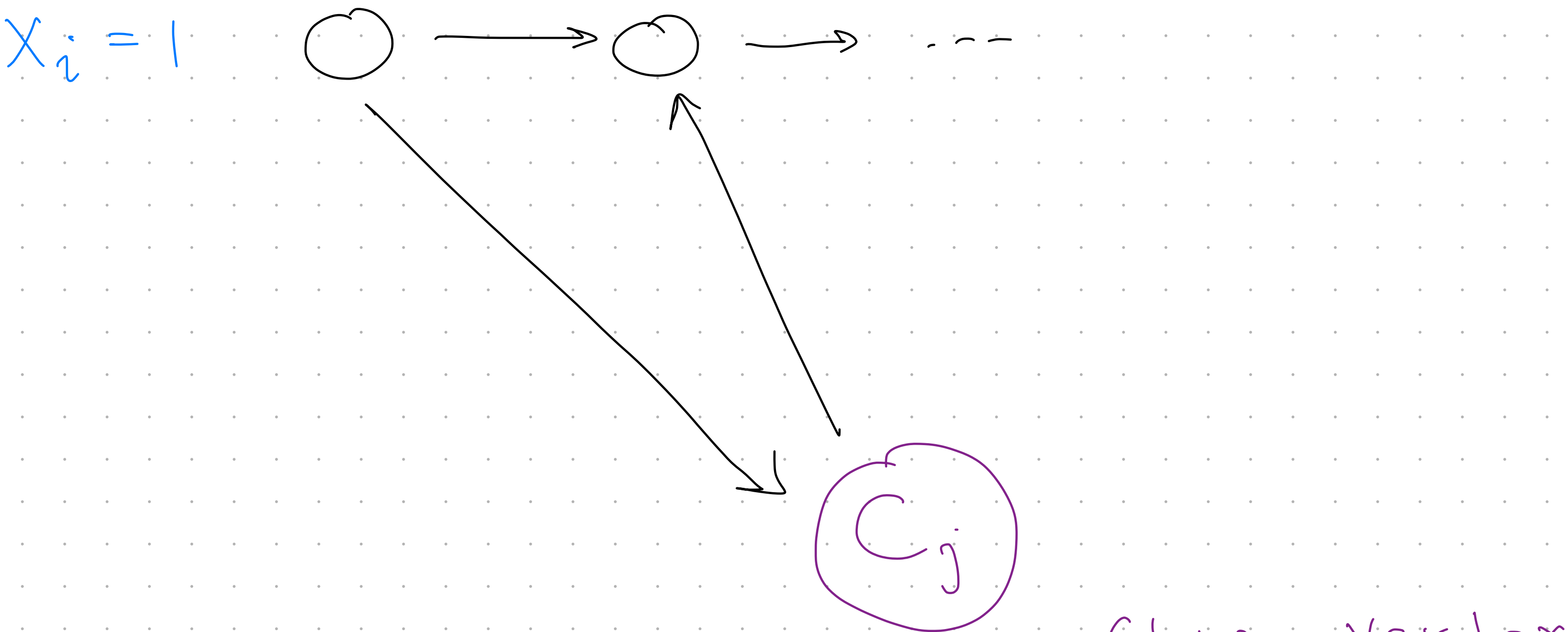
* Clause vertices can only be "visited" if we choose a variable assignment "path" that satisfies clause.

Consider some clause $C_j = (x_i \vee \neg x_j \vee x_u)$

$C_j$

Clause vertex

Consider some clause $C_j = (x_i \vee \neg x_j \vee x_k)$



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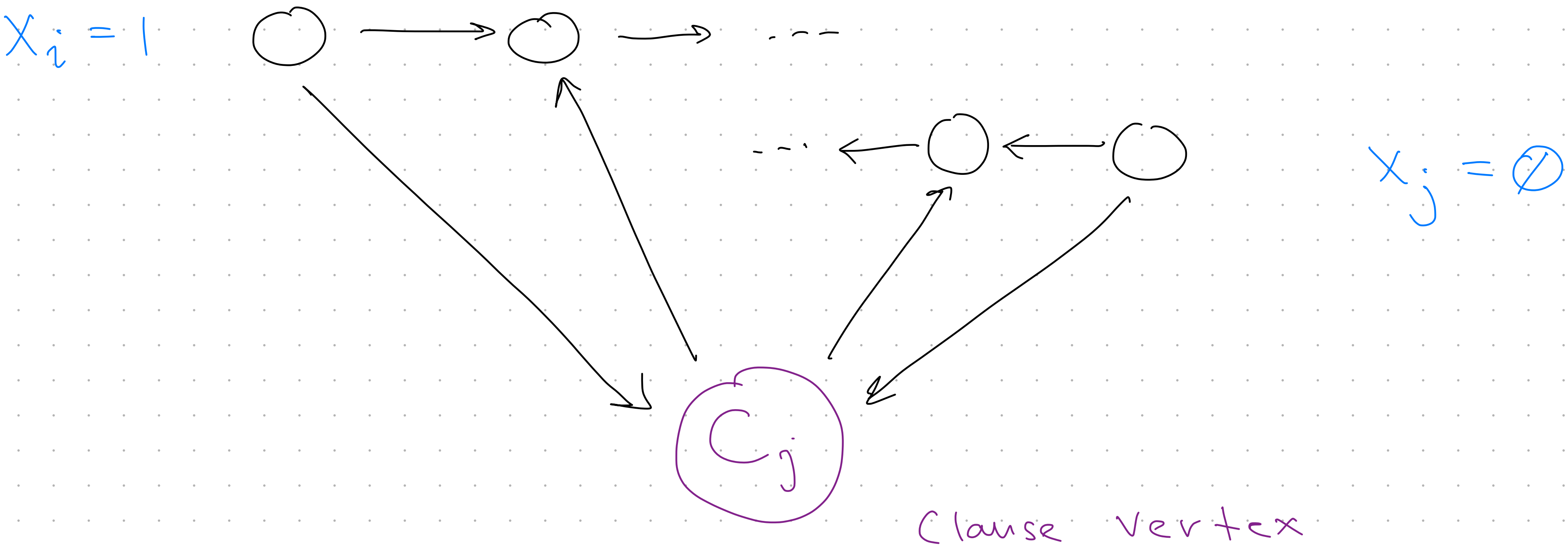
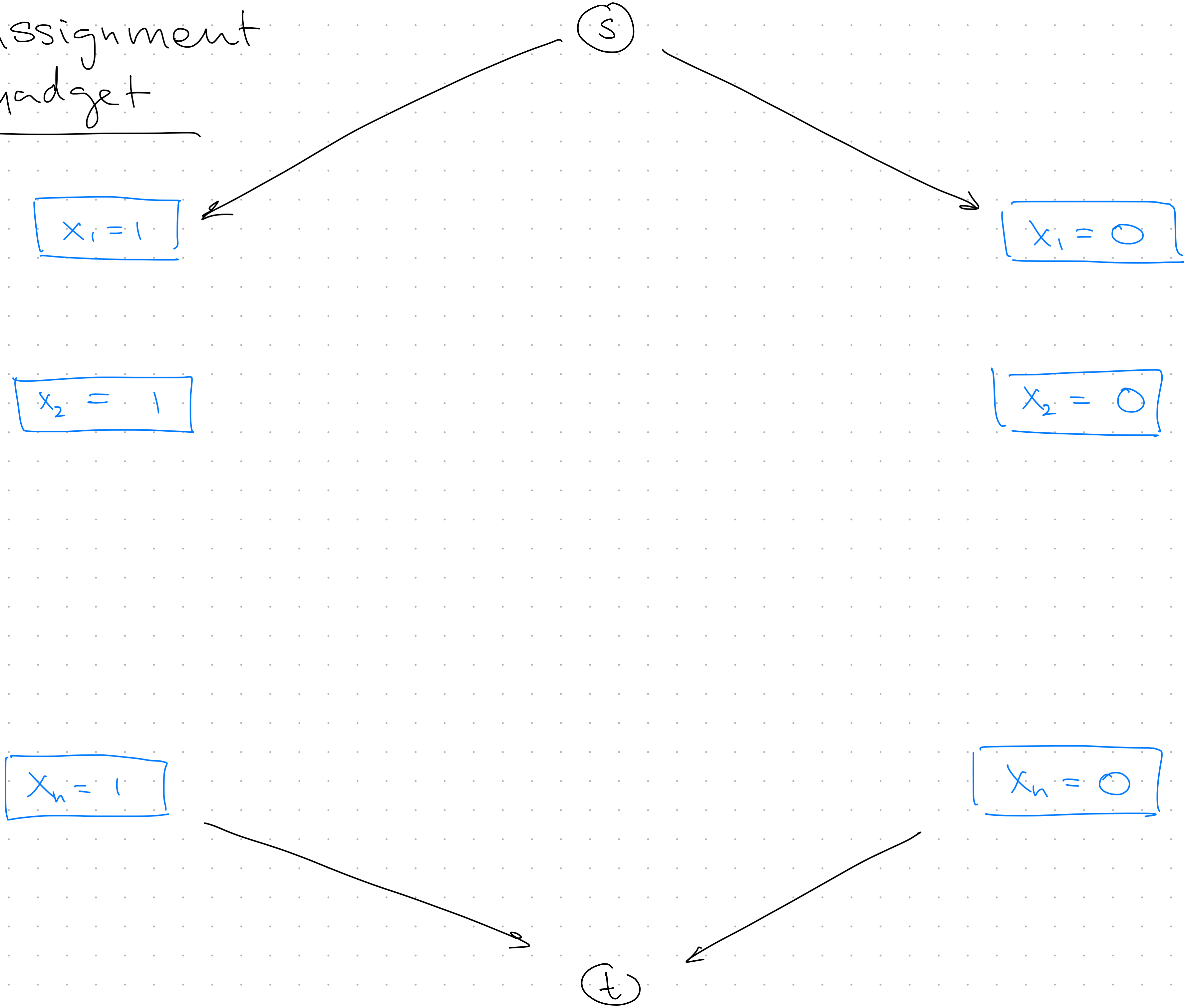



<u>Idea</u>. Direction of path chosen determines assignment of $\vec{x} = \vec{a}$.

# Assignment Gadget



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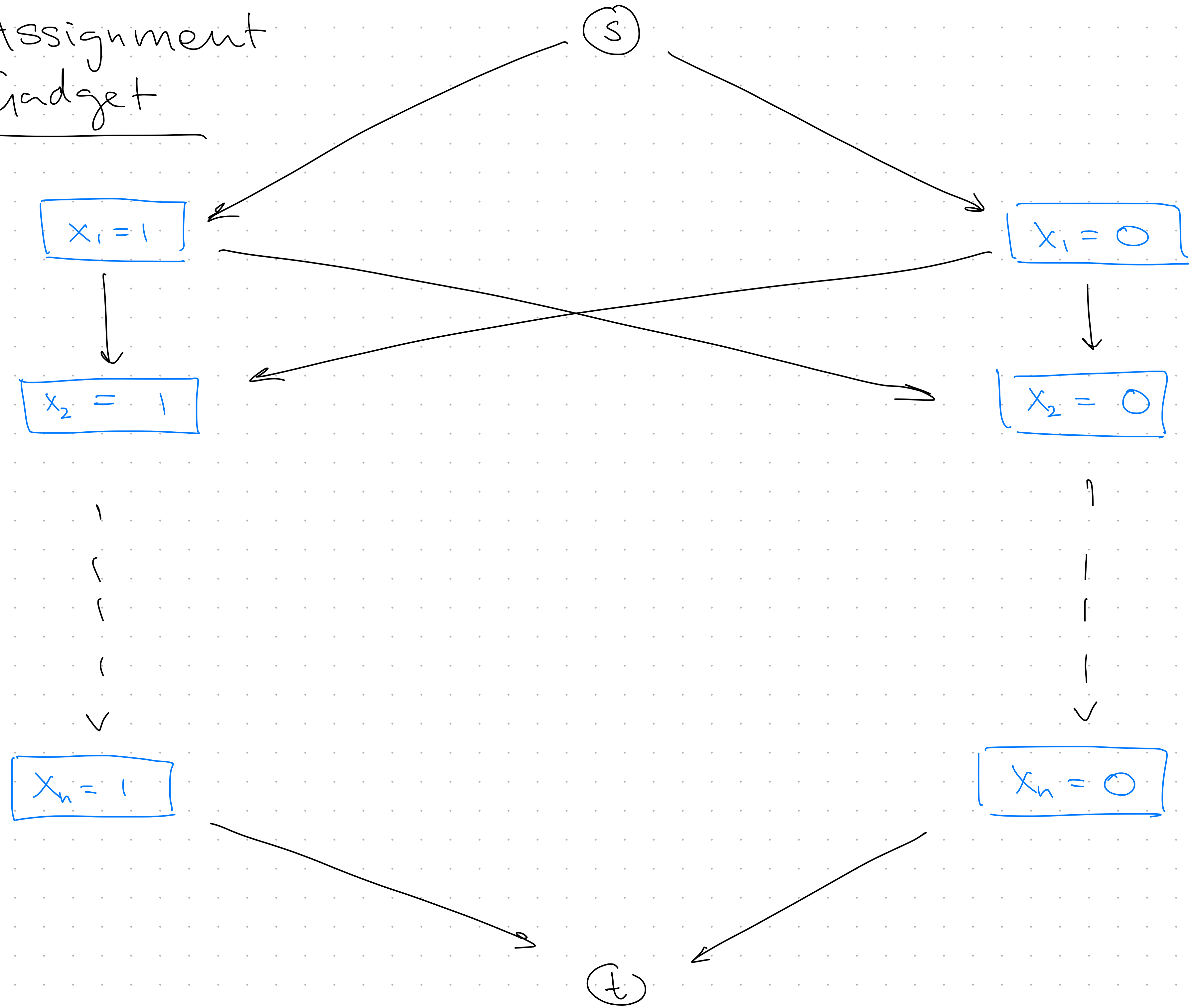
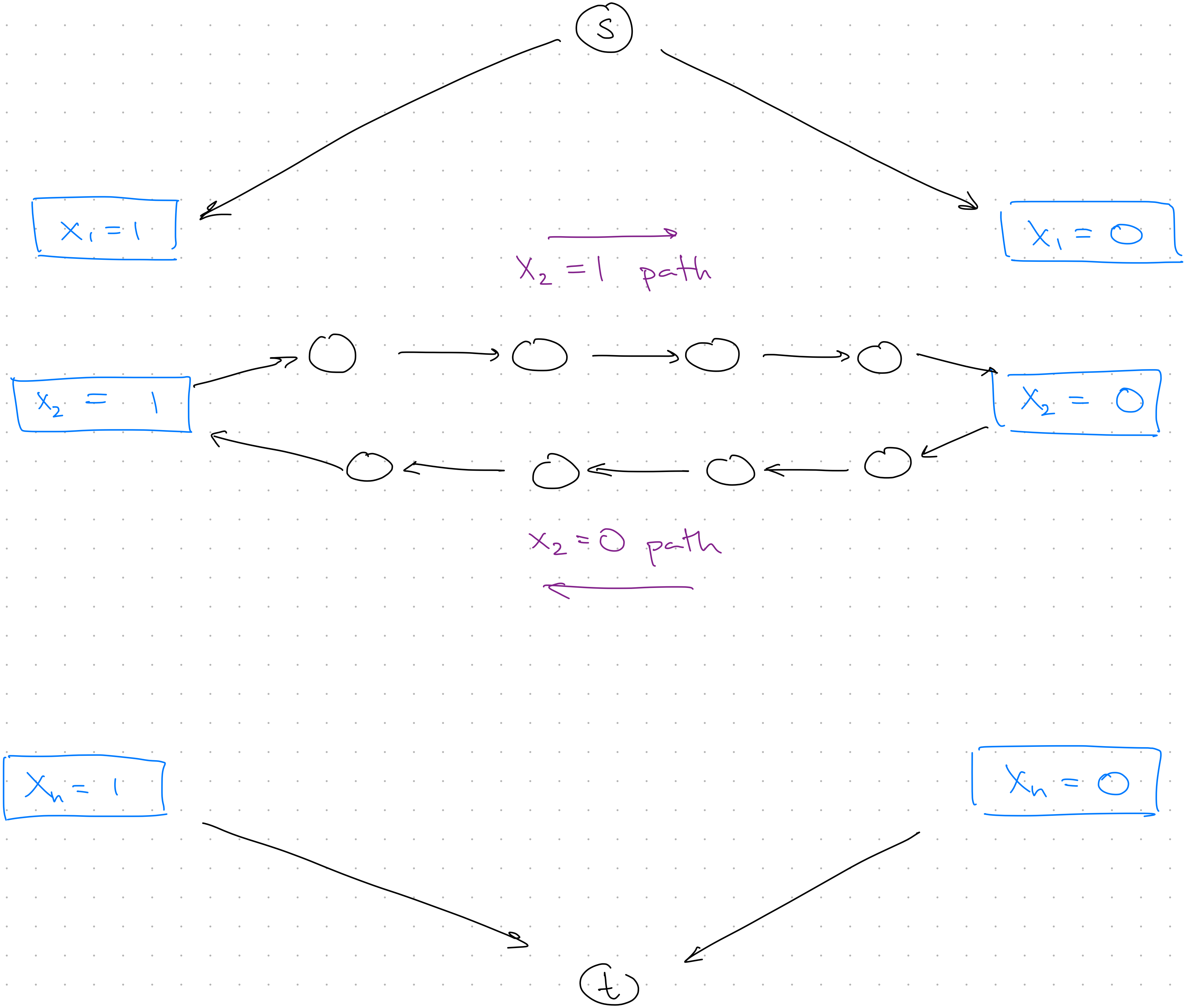
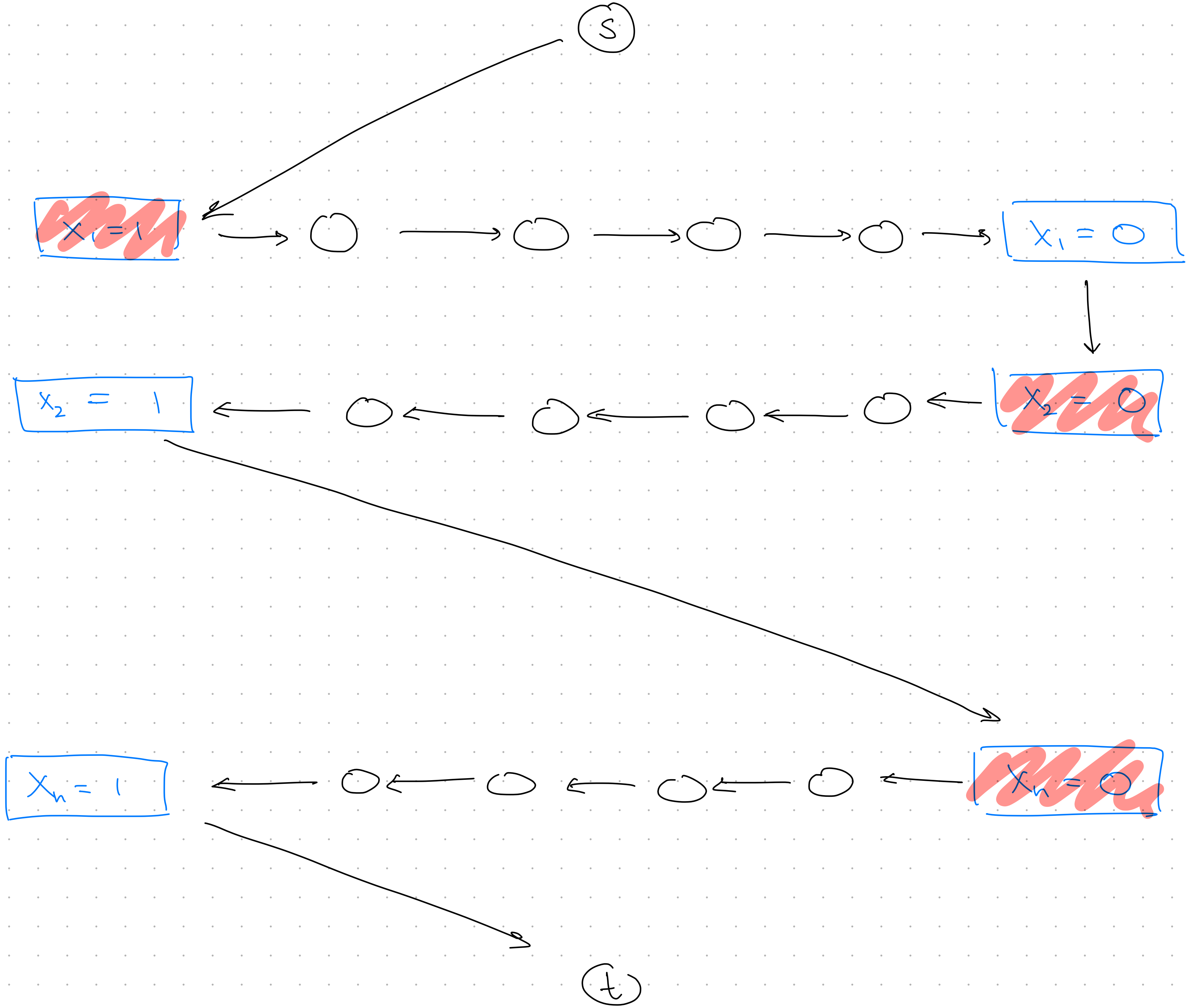



s
x1 = 1
x1 = 0
x2 = 1 path
x2 = 1
x2 = 0
x2 = 0 path
xn = 1
xn = 0
t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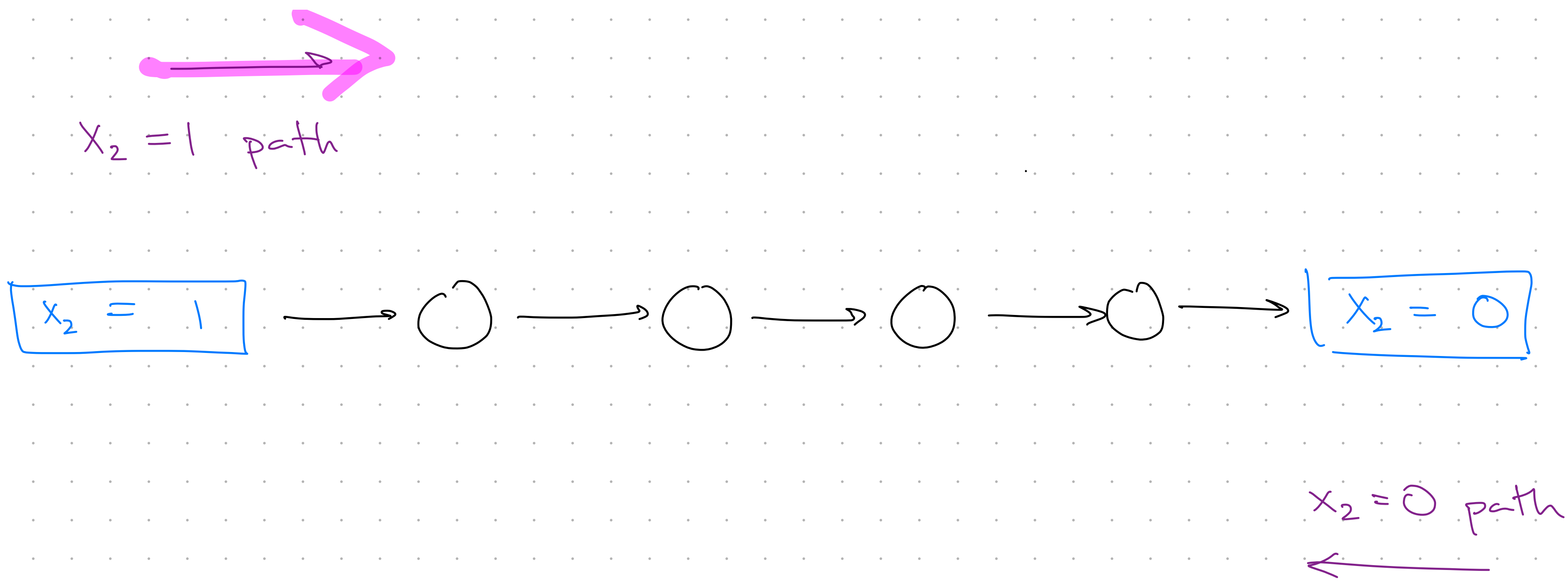

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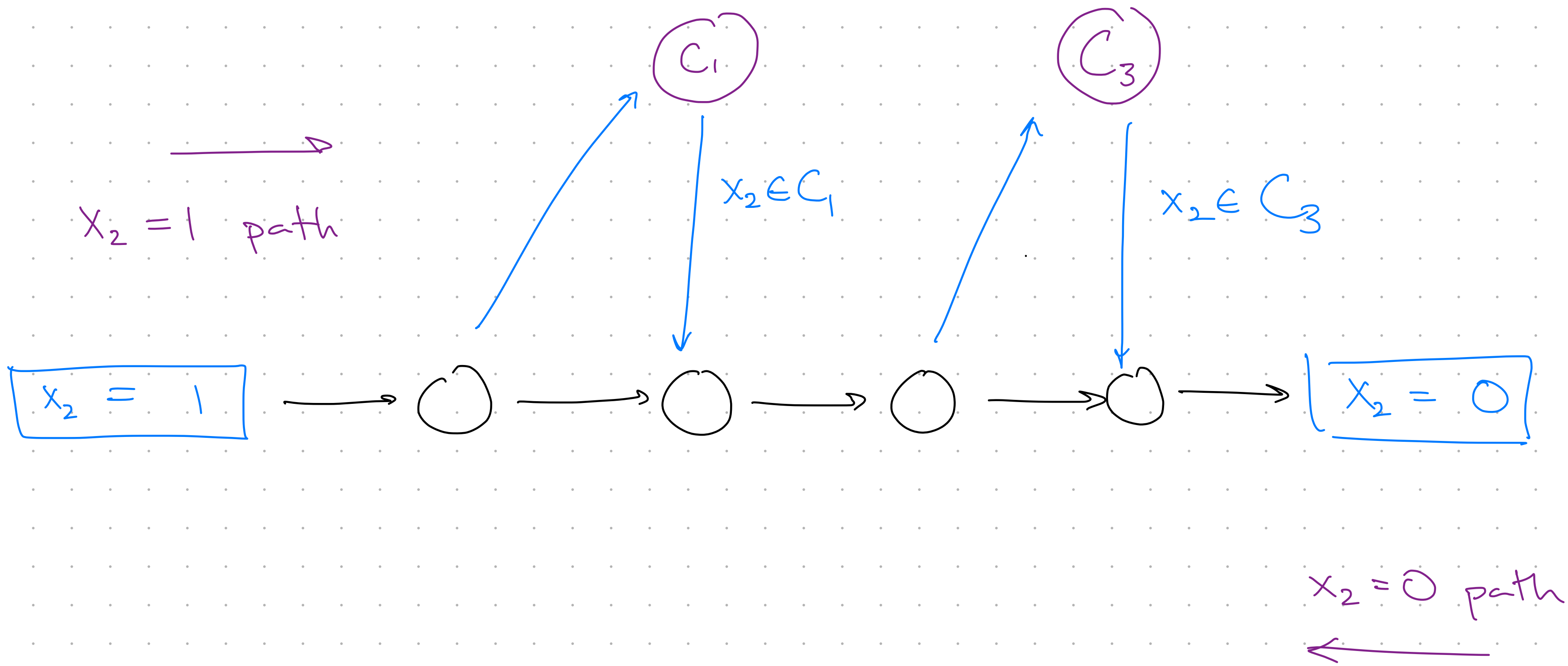

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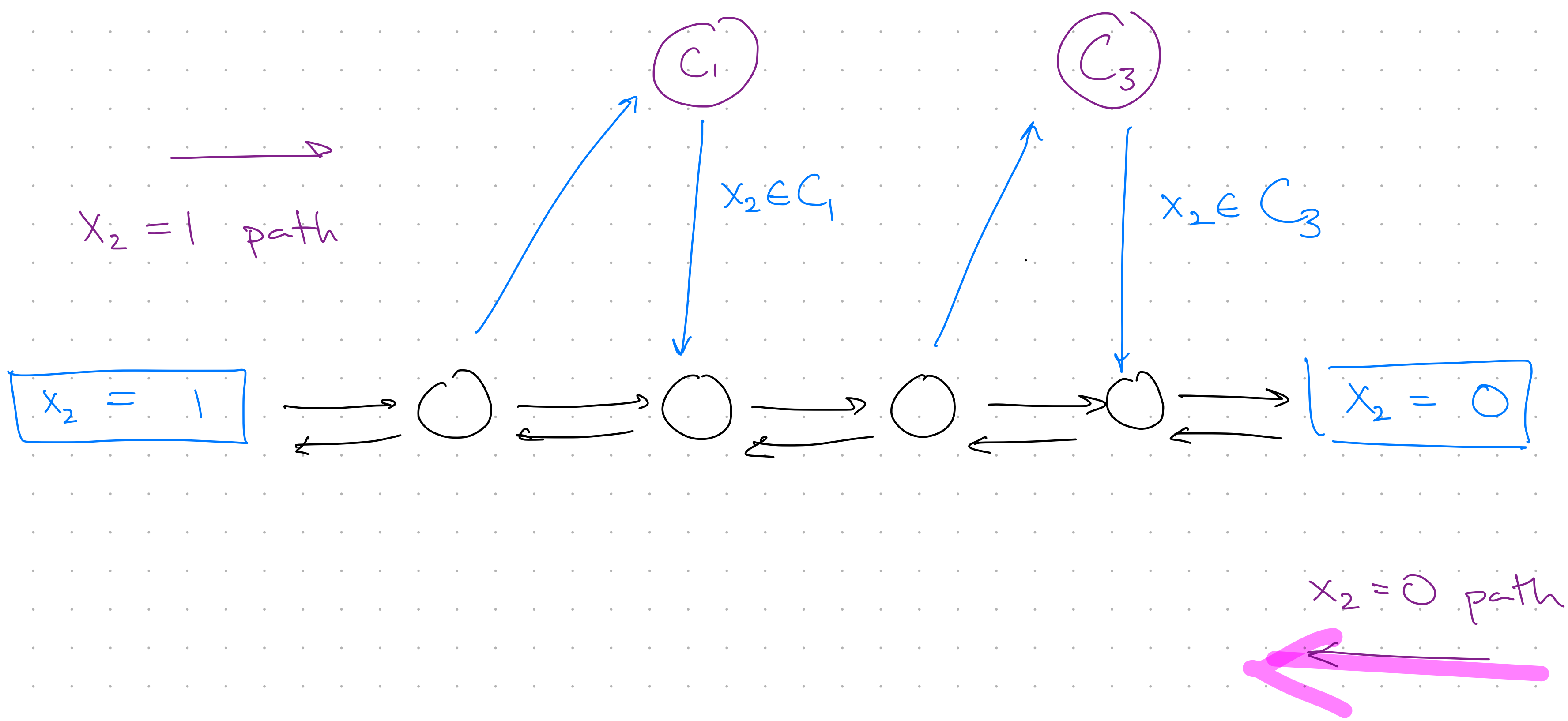

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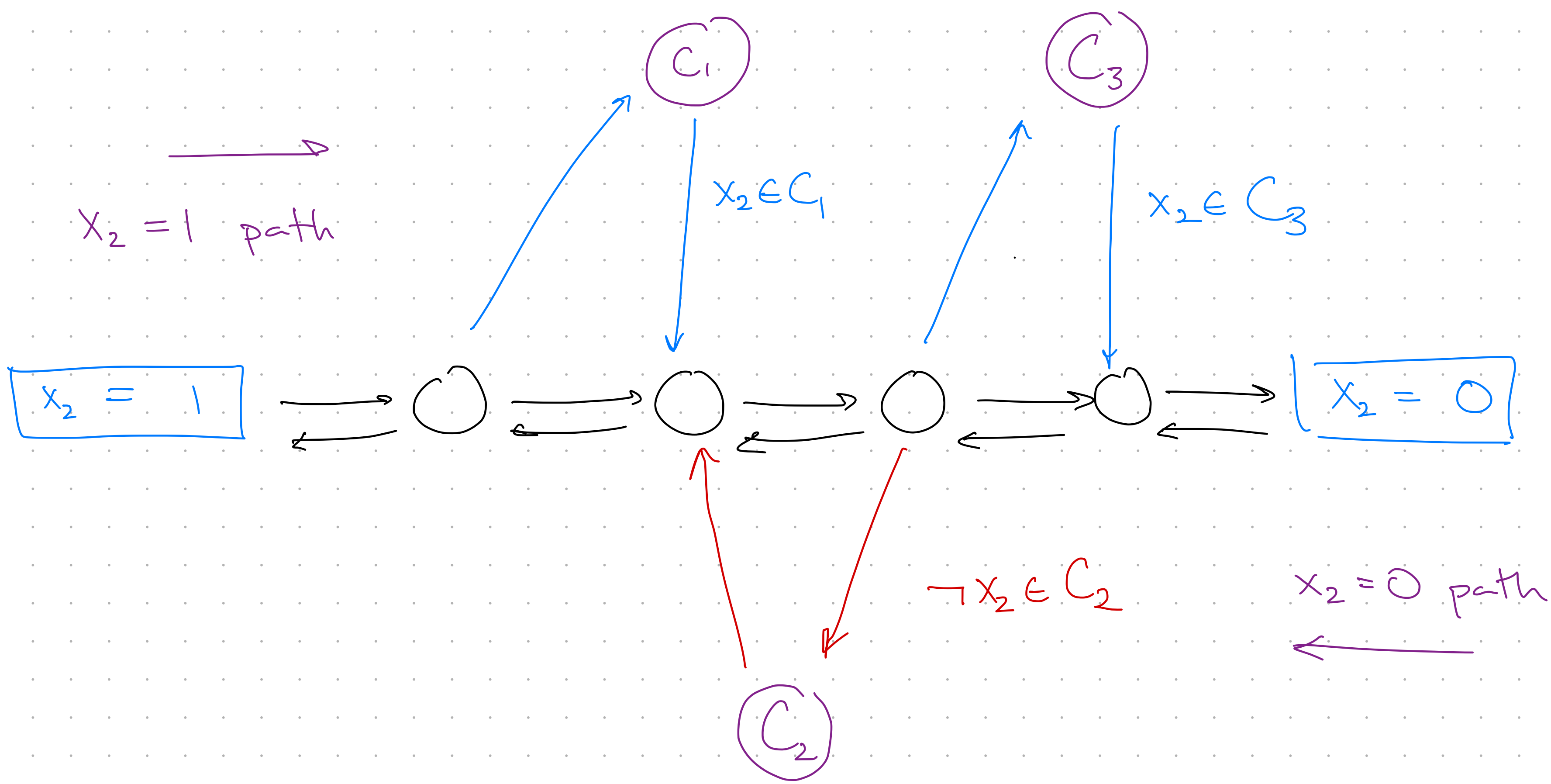

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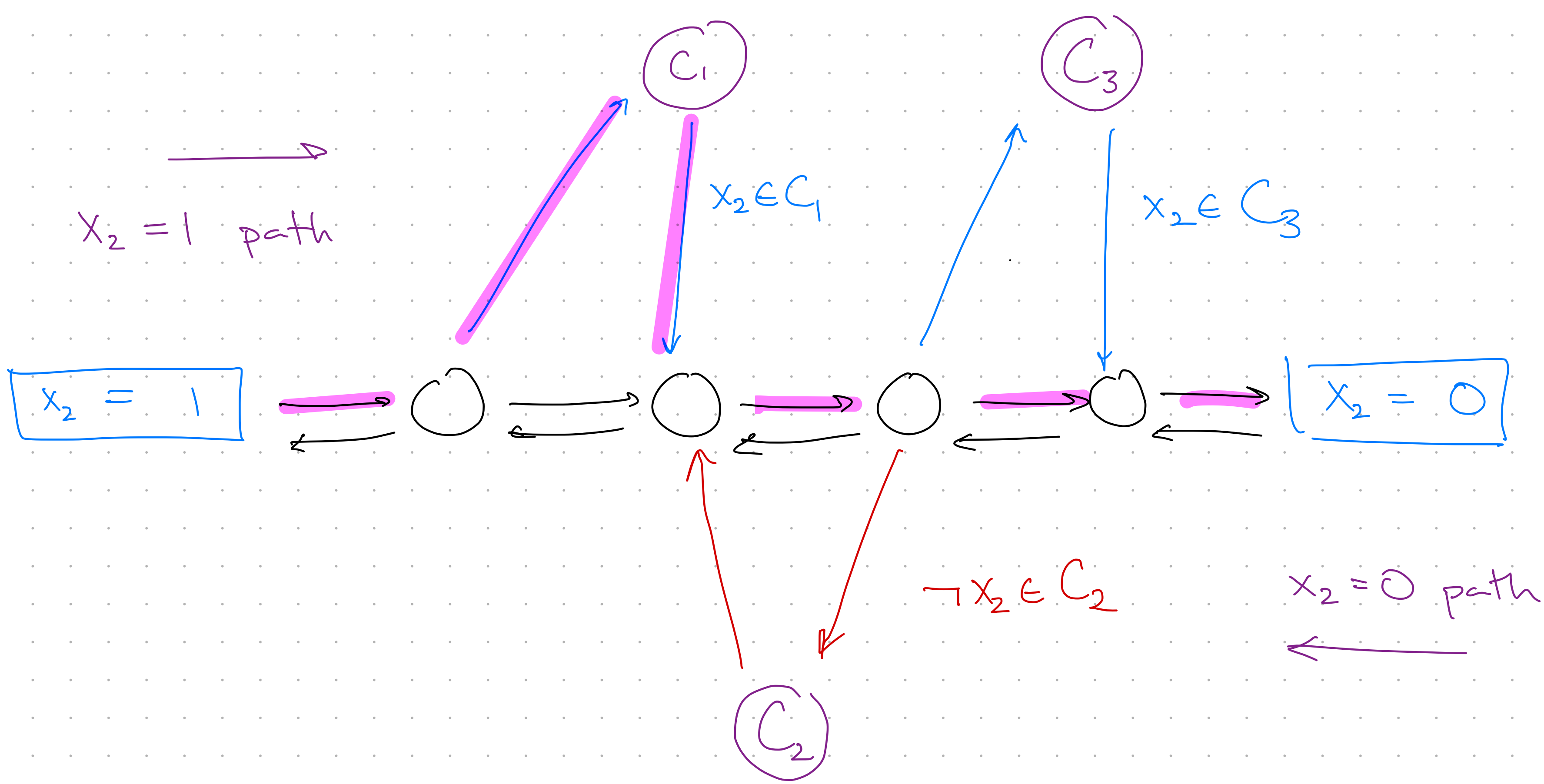
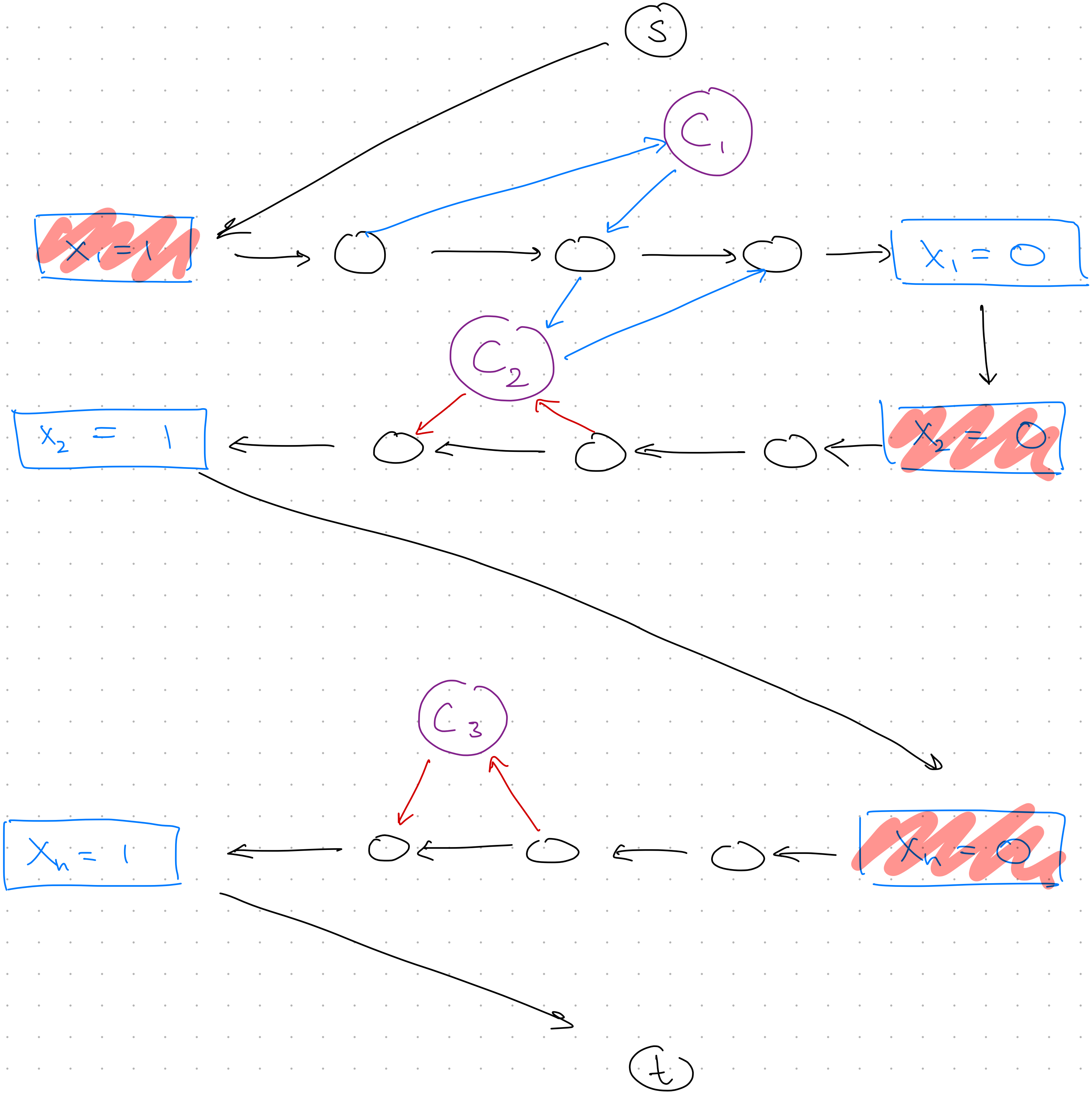



**Claim.** Clause vertex reachable on a path iff path "assigns" variable that satisfies clause



s
C1
x1 = 1
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C2
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C3
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xn = 0
t

Suppose $\phi$ is satisfiable.

* Consider a path through $G$ corresponding to satisfying assignment $\vec{a} \in \{0,1\}^n$

  ↳ For each clause, pick some "representative" literal s.t. literal evaluates to 1 under $\vec{a}$

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For this representative, take the detour to vertex $C_j$, from path associated w/ literal.

Suppose $\phi$ is satisfiable.

* Consider a path through $G$ corresponding to satisfying assignment $\vec{a} \in \{0,1\}^n$

- All variable vertices covered by $\vec{a}$ path

- All clause vertices covered because

$\phi$ satisfied by $\vec{a}$ $\Longrightarrow$ $\exists$ successful detour to $C_j$ for all clauses.

$\Longrightarrow$ Hamiltonian Path in $G$.

$\blacksquare$

Suppose there is a Hamiltonian Path in $G$

* Every simple st-path only selects edges from $\boxed{x_i = 1} \to \boxed{x_i = 0}$ OR $\boxed{x_i = 1} \leftarrow \boxed{x_i = 0}$

↳ Consider an assignment $\bar{a} \in \{0,1\}^n$ based on this orientation.

We argue $\bar{a}$ satisfies $\phi$.

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  ↳ $C_j$ only reachable on paths associated w/ a satisfying literal.

$\Rightarrow$ $\vec{a}$ corresponds to an assignment that satisfies every clause!

$\boxtimes$