19 March 2025 NP-Hard Problems II

Plan

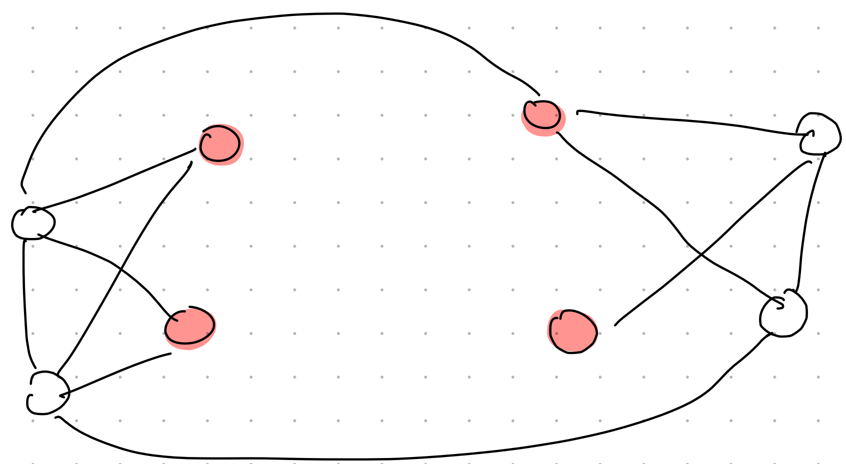
* Independent Set
* Announcements
* More NP-Hard Problems

Maximum Independent Set

Given a graph $G = (V, E)$,

$S \subseteq V$ is an independent set if for all

$$u \in S,\ v \in S \implies (u, v) \notin E$$



$$\text{INDSET} = \{ \langle G, k \rangle : G \text{ contains an independent set of cardinality} \geq k \}$$

Theorem. $3SAT \leq_p \text{INDSET}$.

Corollary. INDSET is NP-HARD.

Reduction from 3SAT to INDSET.

Design polynomial-time algorithm:

* Given ϕ
* Returns $\langle G, k \rangle$

s.t.

ϕ Satisfiable $\iff$ G has IS of cardinality $\geq k$.

$$\phi \in 3SAT \iff \langle G, k \rangle \in INDSET$$

Reduction from 3SAT to INDSET.

Design polynomial-time algorithm:

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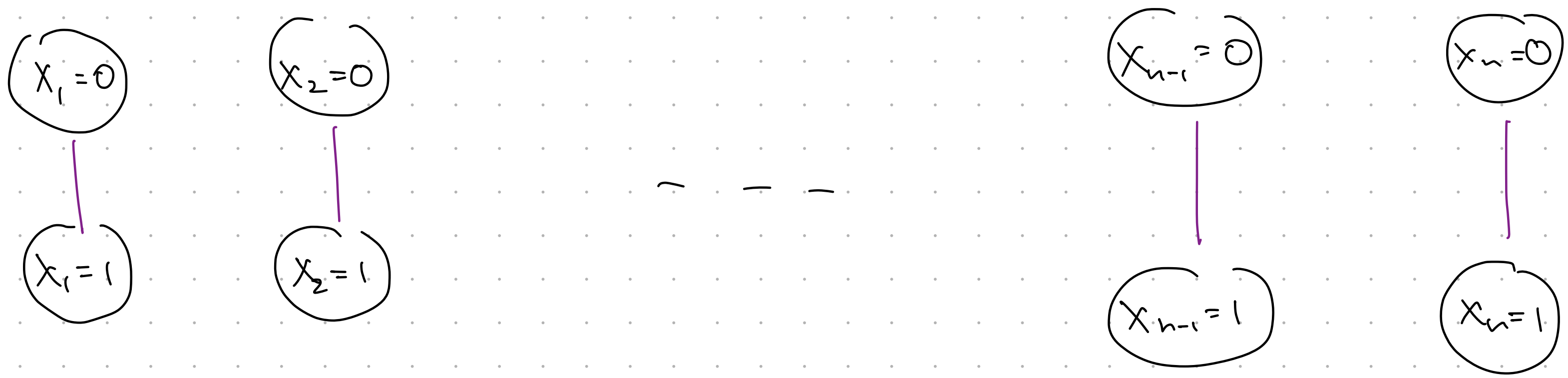
Want. produce a graph where

large IS "selects" assignment that satisfies all clauses.

Variable Assignment Gadgets

For each $i = 1 \cdots n$, $x_i = a_i \in \{0, 1\}$

assigned a single value.



$\forall i = 1, \cdots, n$

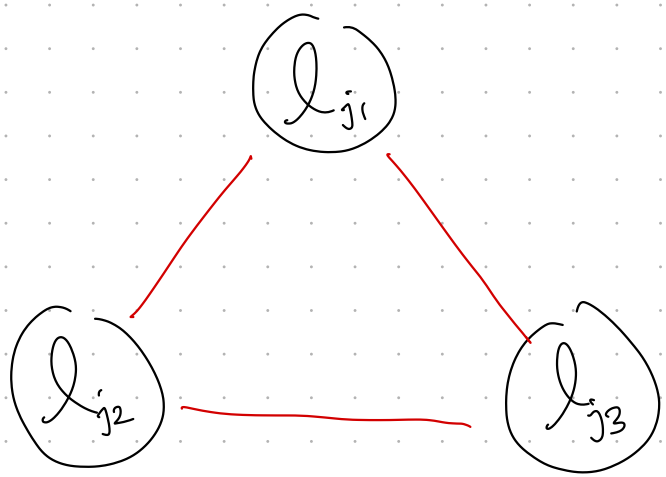
- Create two vertices $x_{i0} = x_i = 0$, $x_{i1} = x_i = 1$ $\in V$.

- Add $(v_{i0}, v_{i1}) \in E$

Clause Satisfaction Gadgets

For each $j = 1, ---, m$ some literal in C_j evaluates to 1
(else, unsatisfiable)

$$C_j = (l_{j1} \vee l_{j2} \vee l_{j3})$$

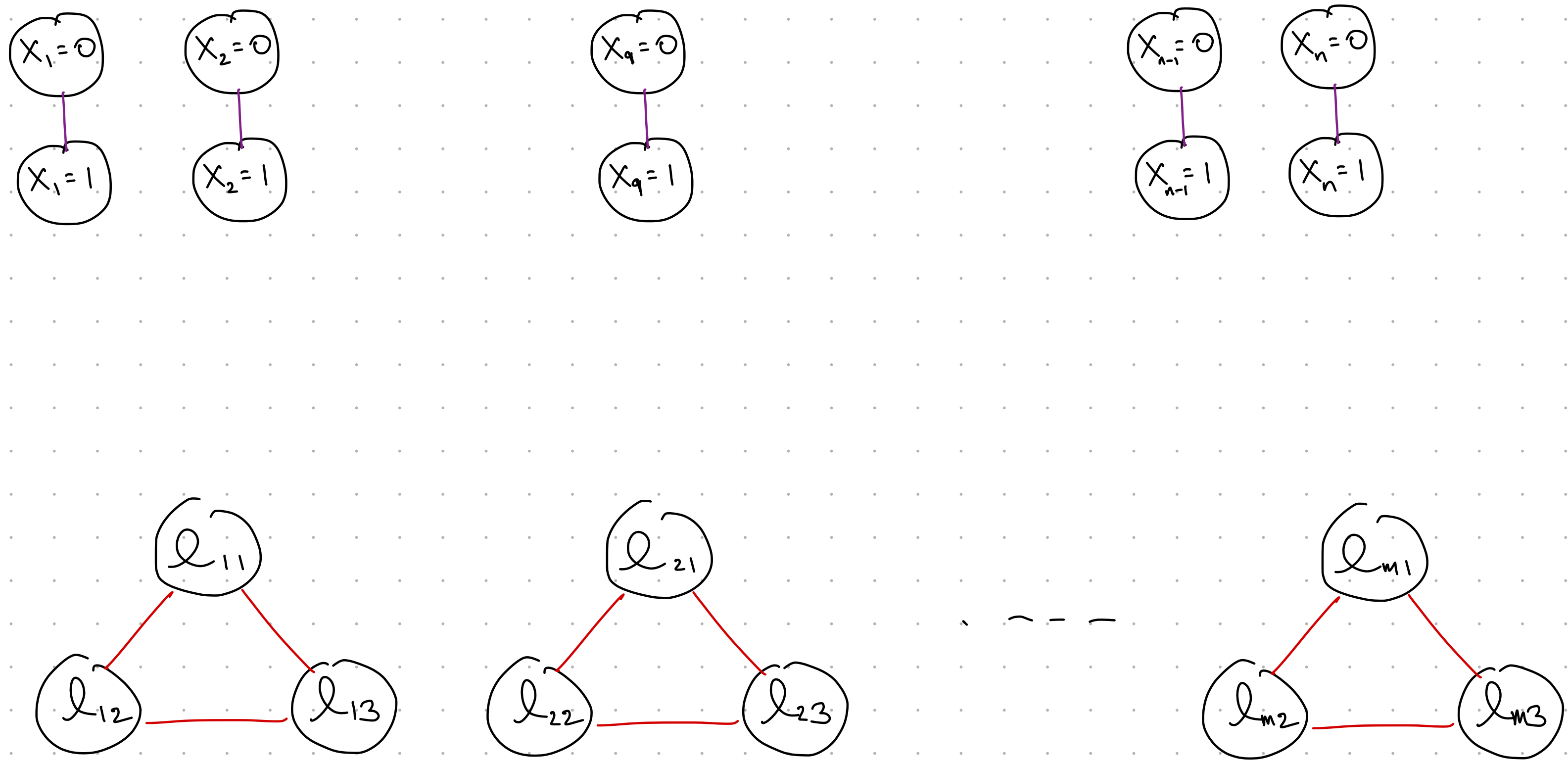


$\forall j = 1 \ldots m$

- Create 3 vertices $l_{j1}, l_{j2}, l_{j3} \in V$

- Add edges between each $(l_{j1}, l_{j2}), (l_{j2}, l_{j3}), (l_{j3}, l_{j1}) \in E$

Clause / Assignment Consistency



$$\phi = (x_1 \vee \neg x_2 \vee x_q) \wedge (\neg x_q \vee x_{n-1} \vee \neg x_n) \wedge \cdots$$

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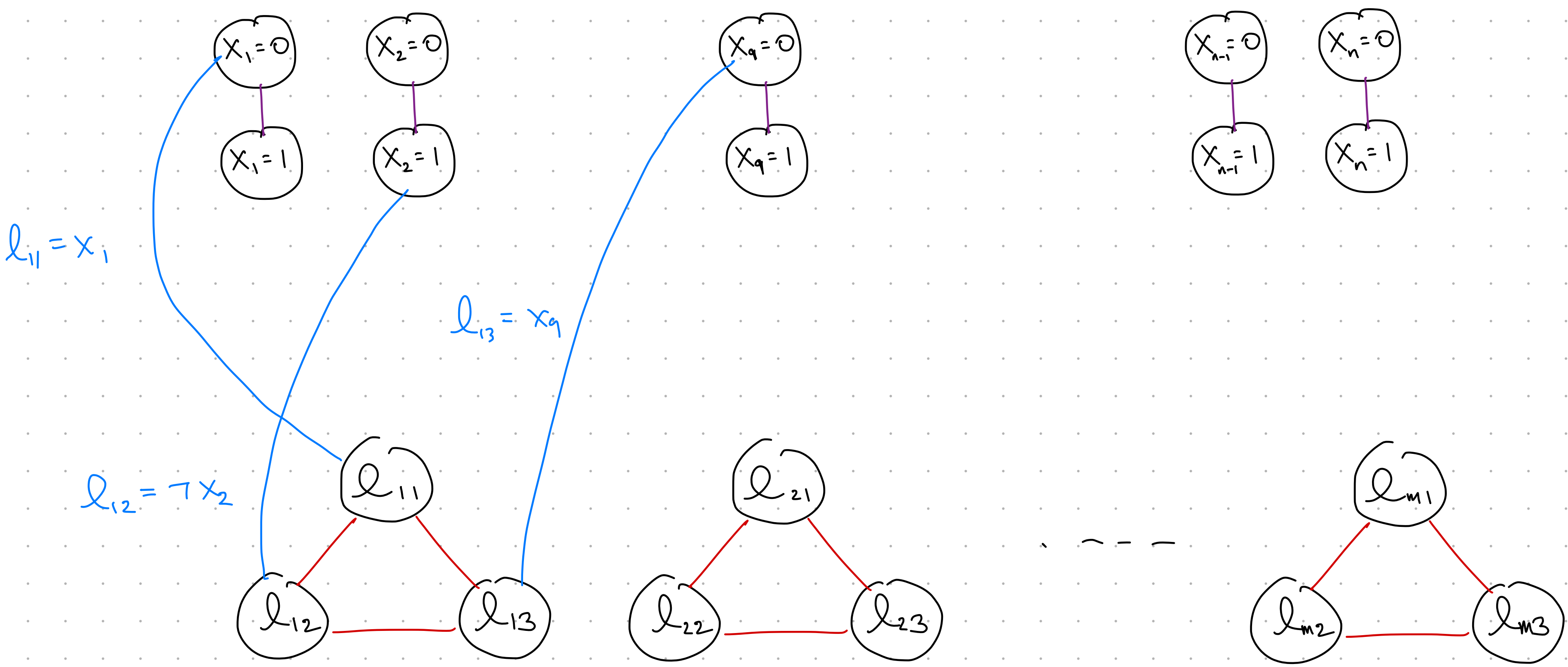
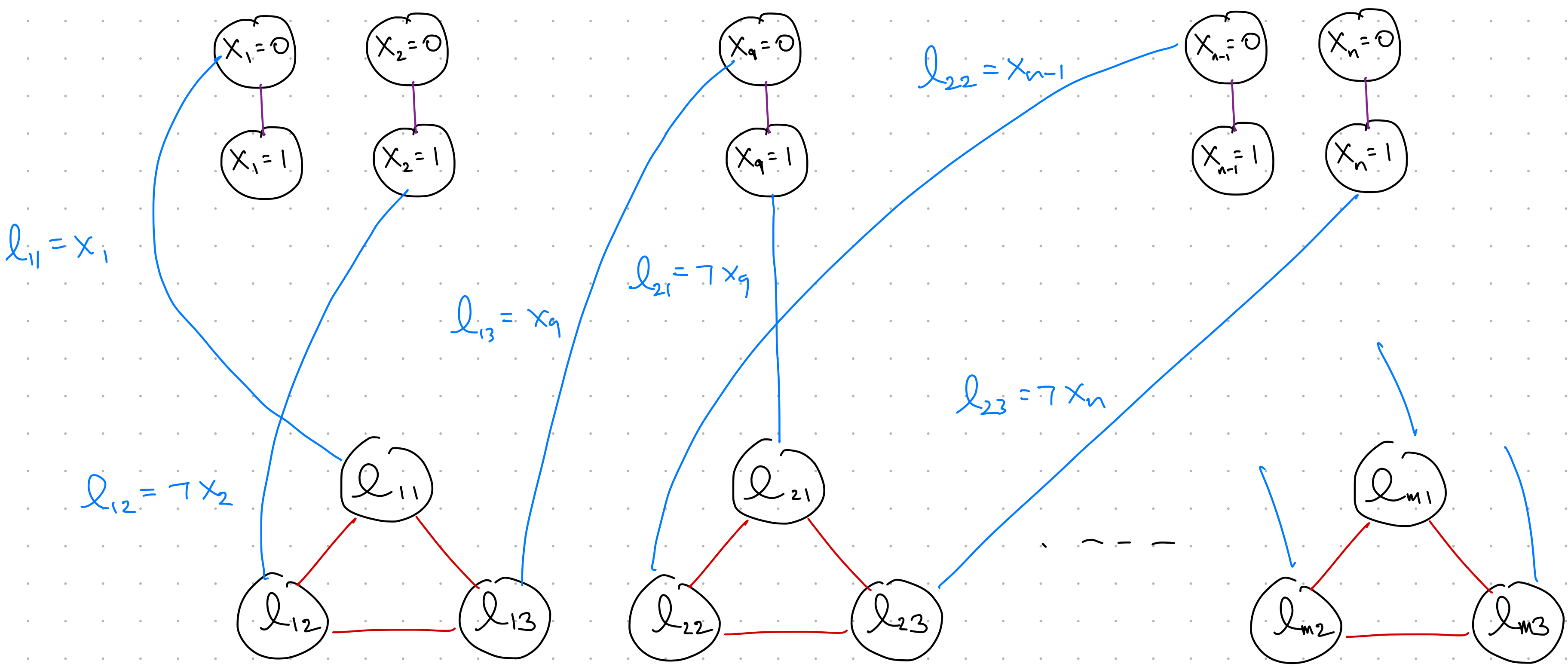


$$\phi = (x_1 \vee \neg x_2 \vee x_q) \wedge (\neg x_q \vee x_{n-1} \vee \neg x_n) \wedge \cdots$$

If literal $l_{jt} = x_i$, add edge $(l_{jt}, x_{i0}) \in E$

If literal $l_{jt} = \neg x_i$, add edge $(l_{jt}, x_{i1}) \in E$

Clause / Assignment Consistency



$$\phi = (x_1 \vee \neg x_2 \vee x_q) \wedge (\neg x_q \vee x_{n-1} \vee \neg x_n) \wedge \cdots$$

For each clause C_j, for each literal l_{jt} for $t=1,2,3$

If literal $l_{jt} = x_i$, add edge $(l_{jt}, x_{i0}) \in E$

If literal $l_{jt} = \neg x_i$, add edge $(l_{jt}, x_{i1}) \in E$

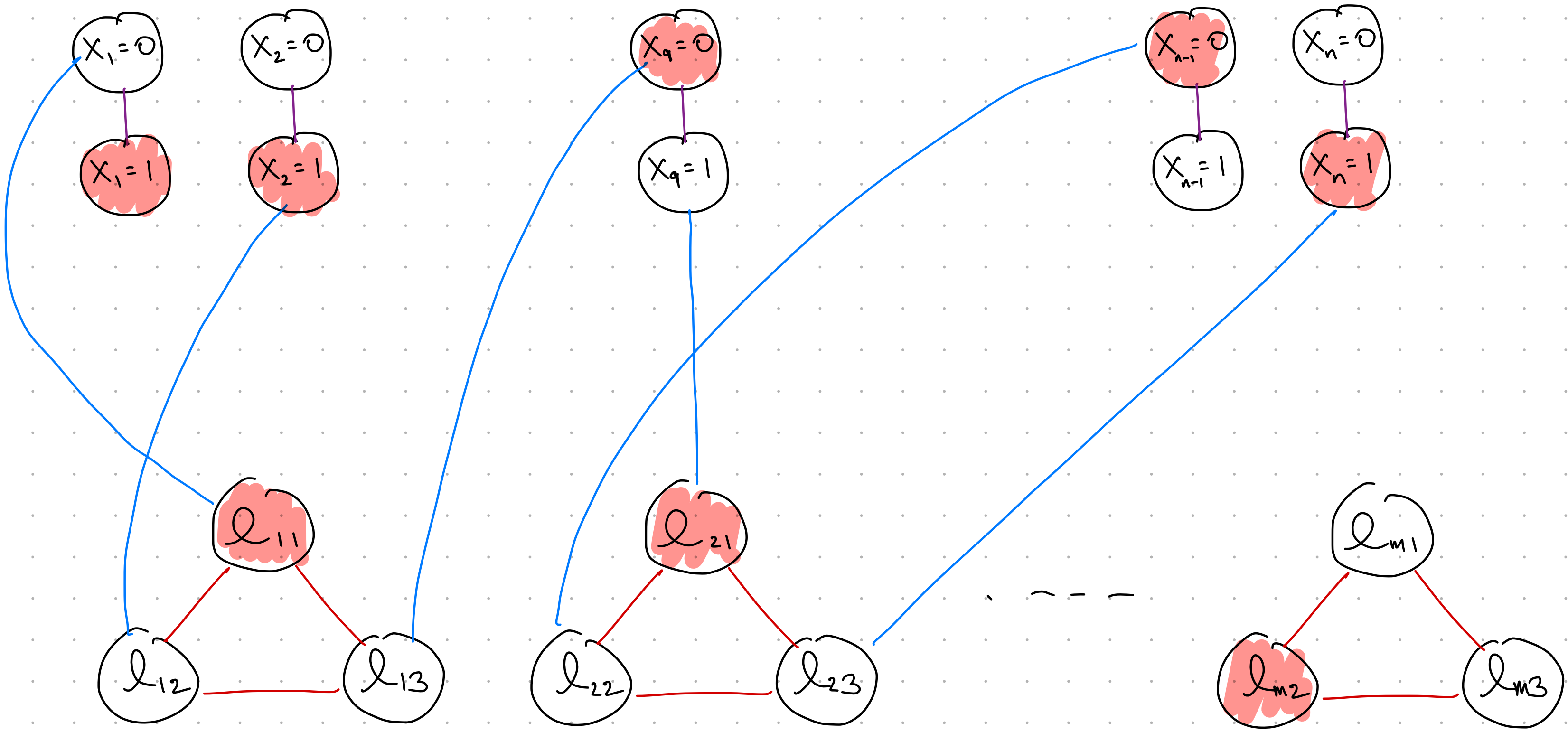
Reduction Running Time

* Construct vertices: 2 for each variable and 3 for each clauses $O(n+m)$

* Construction edges: 1 per variable, 6 per clause $O(n+m)$

$$= O(n+m) = \text{poly}(n,m) \quad \checkmark$$

Clause / Assignment Consistency



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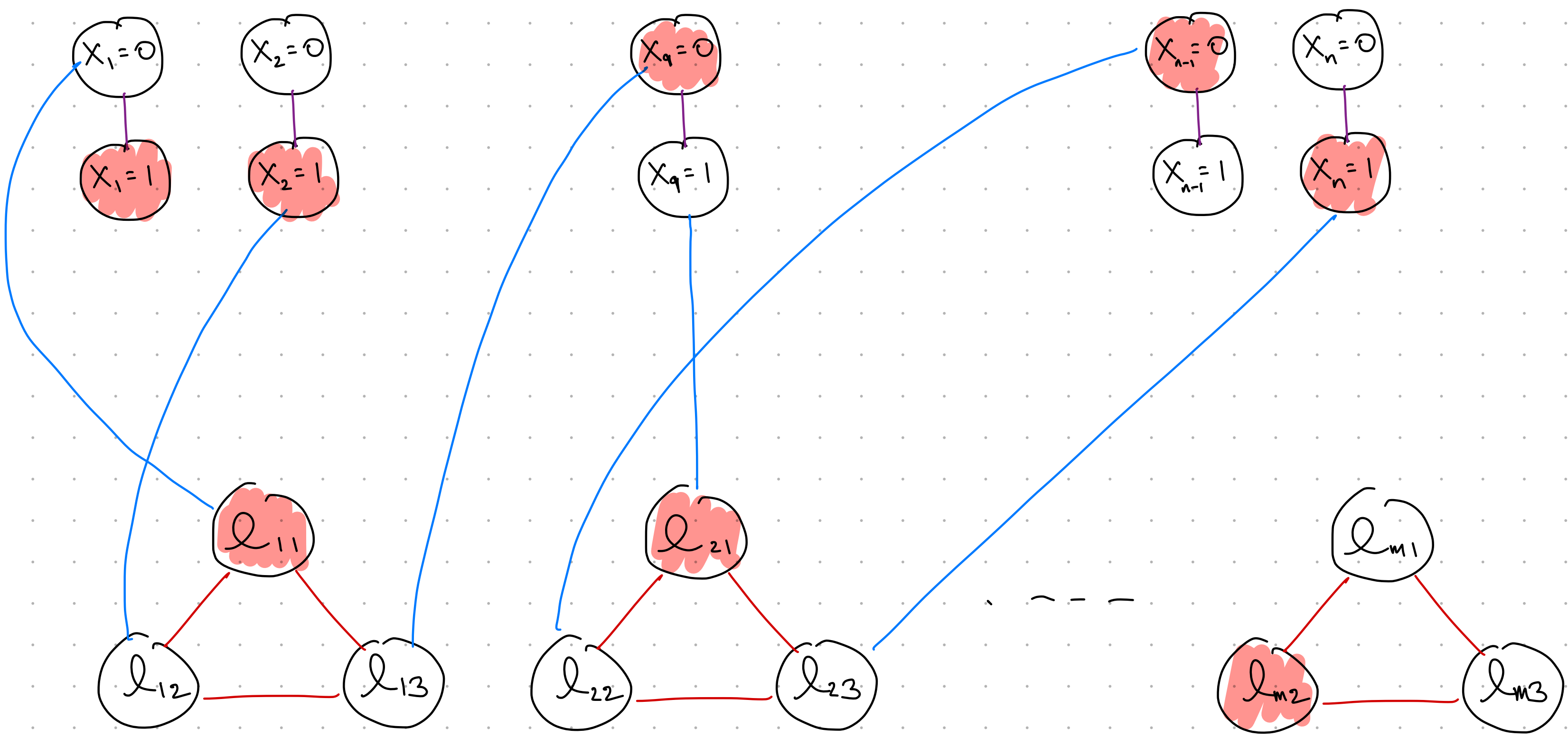
Claim. ϕ is satisfiable $\iff$ G has an IS of cardinality $\geq n+m$.

Announcements

* HW5 due
* HW6 released today
* Prelim 2 : Next Thursday
  ↦ Practice Questions Released Today

Clause / Assignment Consistency



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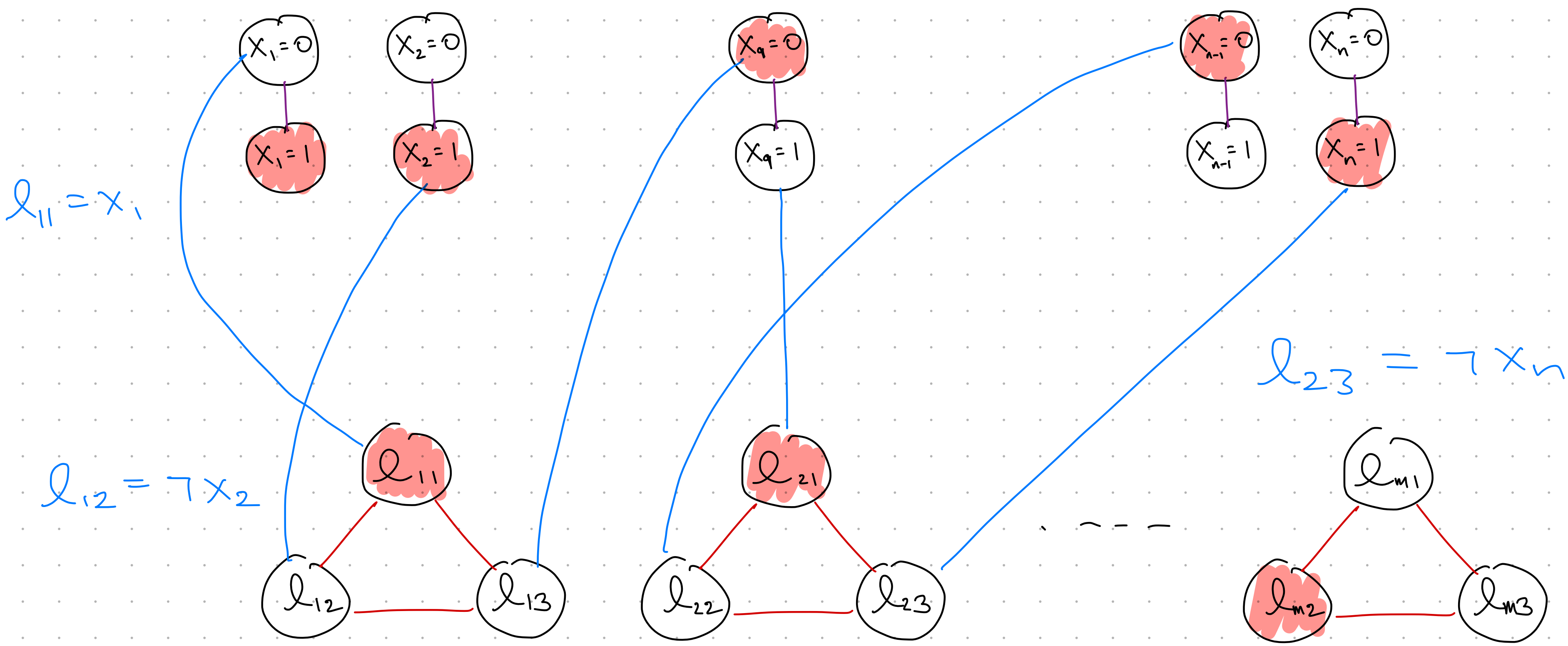


$$\phi = (x_1 \vee \neg x_2 \vee x_q) \wedge (\neg x_q \vee x_{n-1} \vee \neg x_n) \wedge \cdots$$

Key Property

$$(\ell_{jt}, x_i b) \in E \iff x_i = b \Rightarrow \ell_{jt} = 0$$

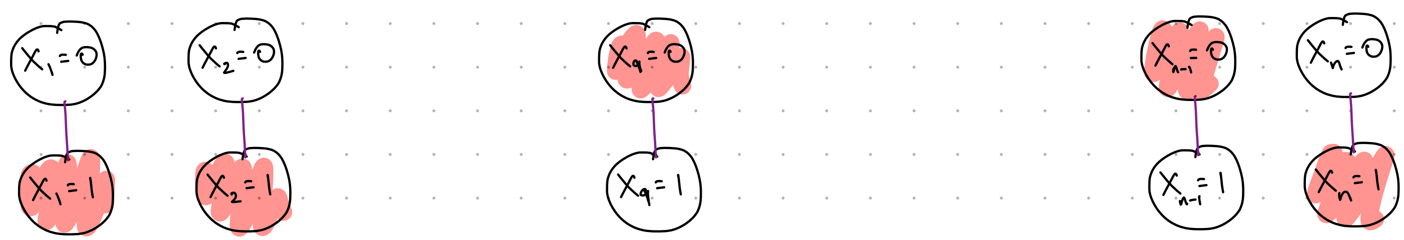
ϕ satisfiable $\Longrightarrow$ IS of size $n+m$.

* Consider a satisfying assignment $\vec{a} \in \{0,1\}^n$.

ϕ satisfiable $\implies$ IS of size $n+m$.

* Consider a satisfying assignment $\vec{a} \in \{0,1\}^n$.
* For each $i = 1 \dots n$, add x_{ib} to S for $a_i = b$.

$a_1 = 1 \quad a_2 = 1 \quad \dots \quad a_q = 0 \quad \dots \quad a_{n-1} = 0 \quad a_n = 1$



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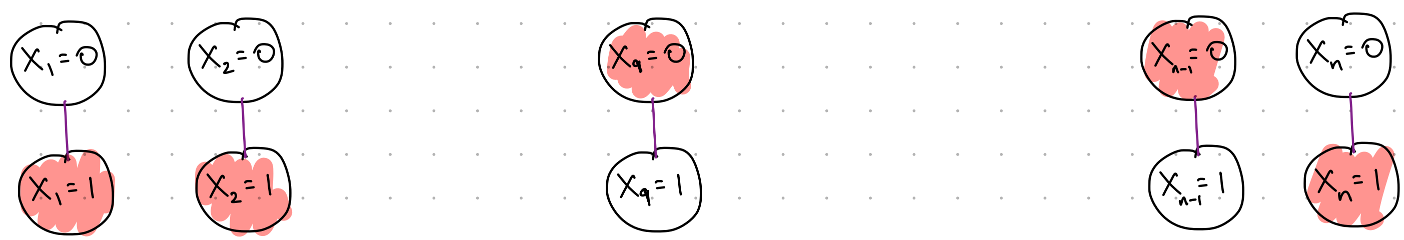
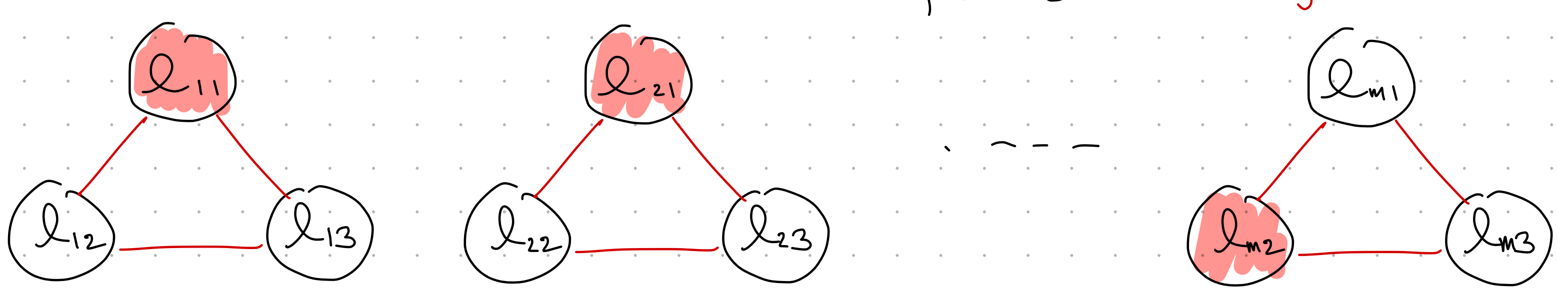


* For each $j = 1 \ldots m$, add literal ℓ_{jt} to S for some $\ell_{jt} = 1$ under $\vec{a}$.



↳ Each clause chooses some representative $\ell_{jt} = 1$ under $\vec{x} = \vec{a}$.

ϕ satisfiable $\Longrightarrow$ IS of size $n+m$.

* Consider a satisfying assignment $\vec{a} \in \{0,1\}^n$.
* For each $i = 1 \ldots n$, add x_{ib} to S for $a_i = b$.
* For each $j = 1 \ldots m$, add literal l_{jt} to S for some $l_{jt} = 1$ under $\vec{a}$.

Claim 0. $|S| = n+m$

Claim 1. S is an independent set.

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Claim 0. $|S| = n+m$

Claim 1. S is an independent set.

* only select 1 vertex per variable gadget
  1 vertex per clause gadget.
* $(l_{jt}, x_{ib}) \in E$ only if setting $x_i = b$ causes $l_{jt} = 0$.
  $\Longrightarrow (l_{jt}, x_{ib}) \notin E$.

$\blacksquare$

Φ unsatisfiable $\Longrightarrow$ Every IS has cardinality $< n+m$

---

* Every IS has $\leq n$ vertices from variable gadgets
  & $\leq m$ vertices from clause gadgets

ϕ unsatisfiable $\Longrightarrow$ Every IS has cardinality $< n+m$

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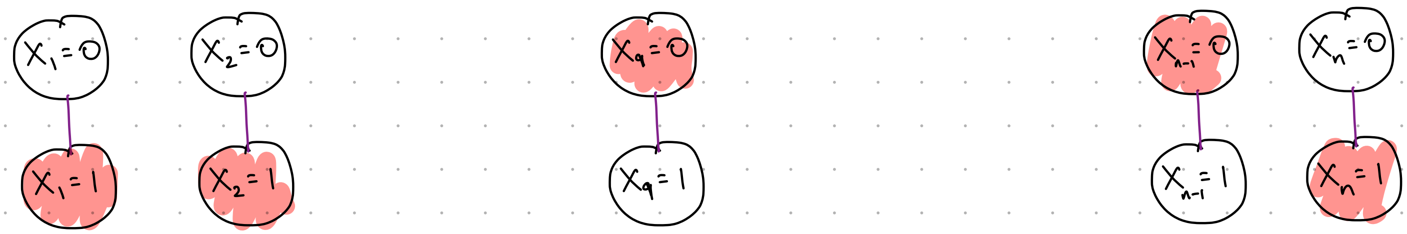
Imagine selecting n vertices from variable gadgets



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* Every IS has $\leq n$ vertices from variable gadgets
  & $\leq m$ vertices from clause gadgets

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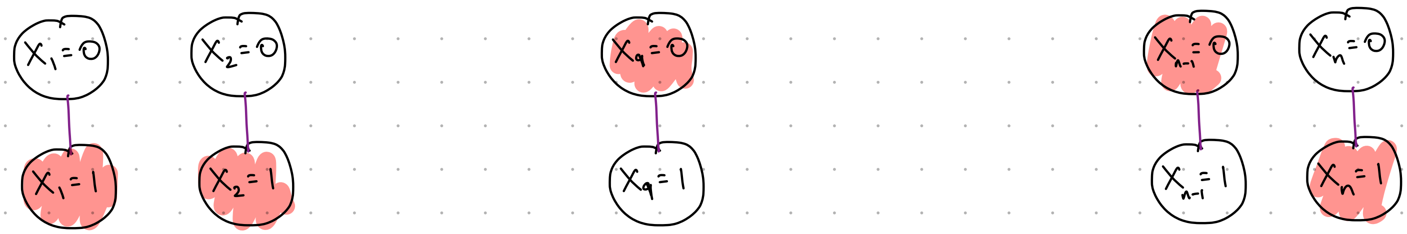
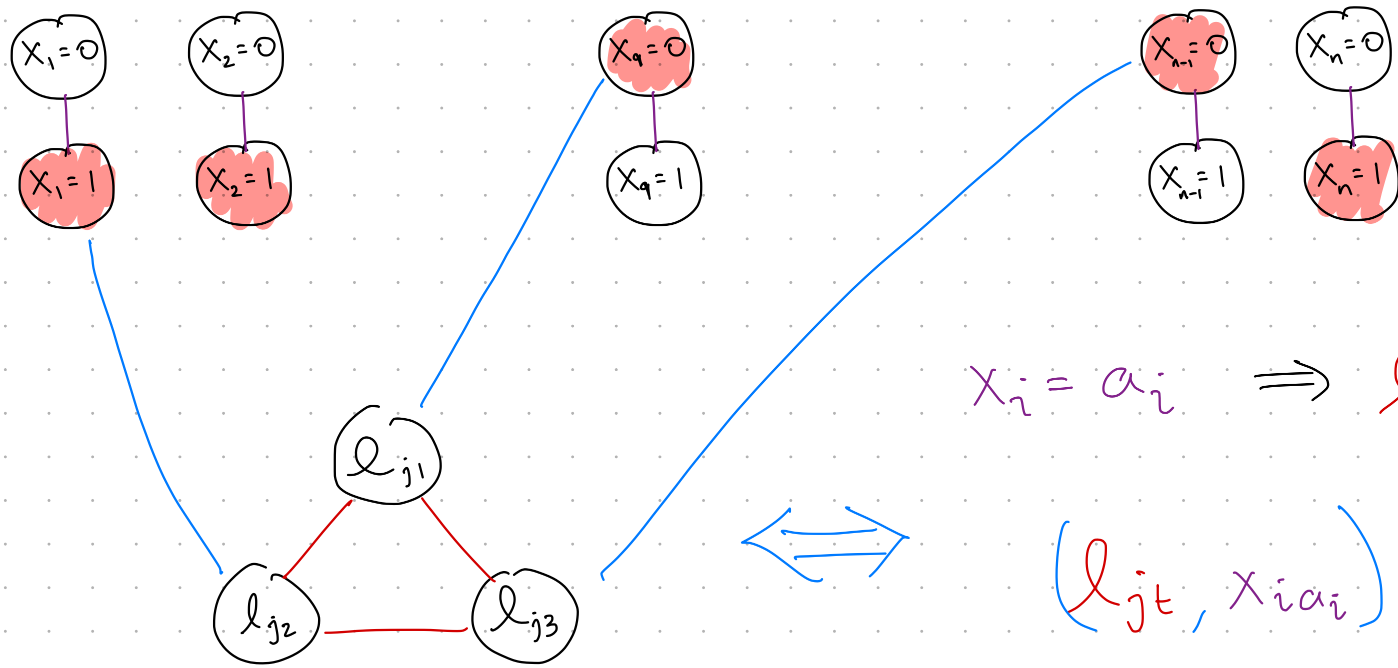


Such a selection corresponds to some $\vec{a} \in \{0,1\}^n$

* Every assignment $\vec{a} \in \{0,1\}^n$ results in some clause C_j with every literal $\ell_{j_1} = \ell_{j_2} = \ell_{j_3} = 0$.

Φ unsatisfiable $\Longrightarrow$ Every IS has cardinality $< n+m$

Imagine selecting n vertices from variable gadgets



$x_i = a_i \Longrightarrow l_{jt} = 0$

$\Longleftrightarrow (l_{jt}, x_{ia_i}) \in E$

* Every assignment $\vec{a} \in \{0,1\}^n$ results in some clause C_j with every literal $l_{j_1} = l_{j_2} = l_{j_3} = 0$.

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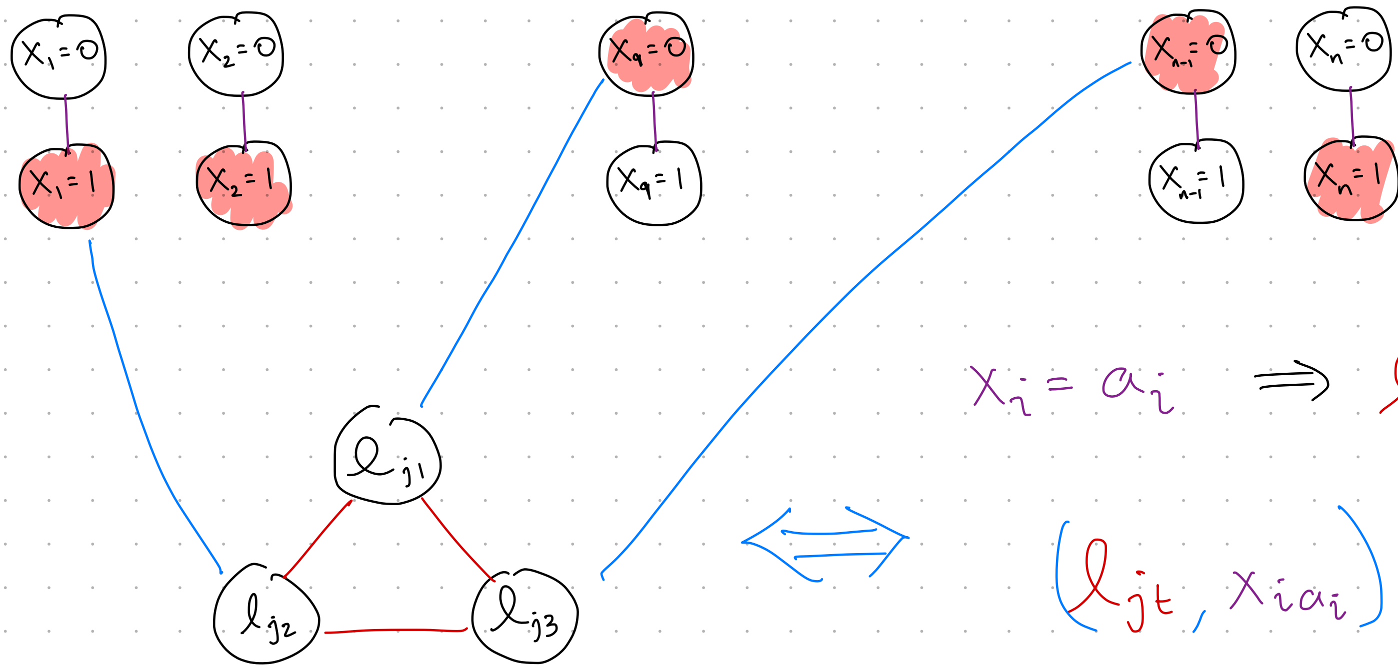
Imagine selecting n vertices from variable gadgets



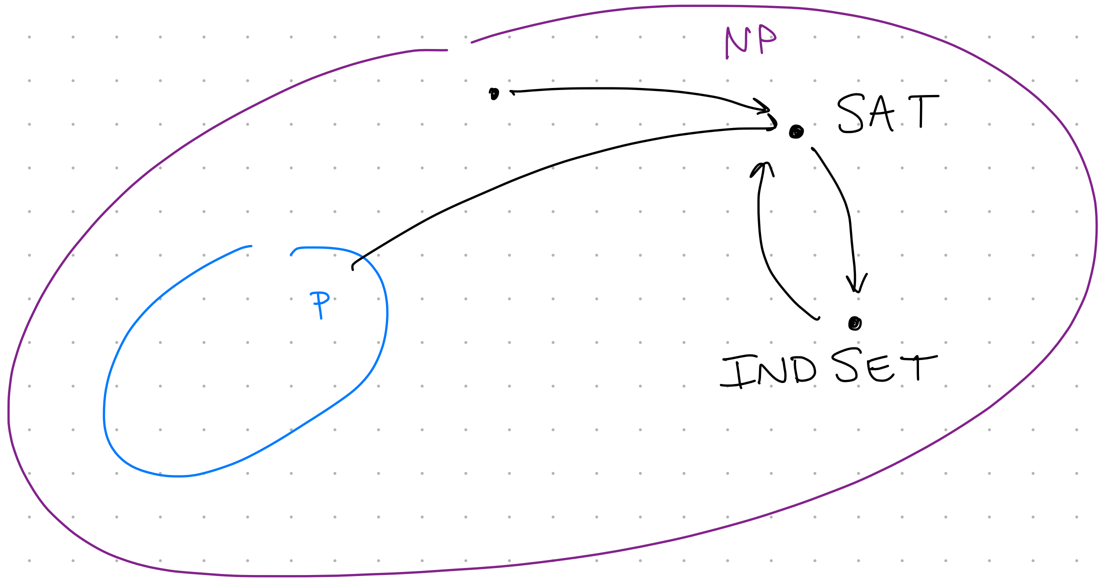
$x_i = a_i \Longrightarrow \ell_{jt} = 0$

$\Longleftrightarrow (\ell_{jt}, x_{ia_i}) \in E$

* Every ℓ_{jt} has some neighbor already selected in S

Thus, IS cannot have 1 vertex per variable
and 1 vertex per clause.

$\Longrightarrow$ IS $< n+m$.



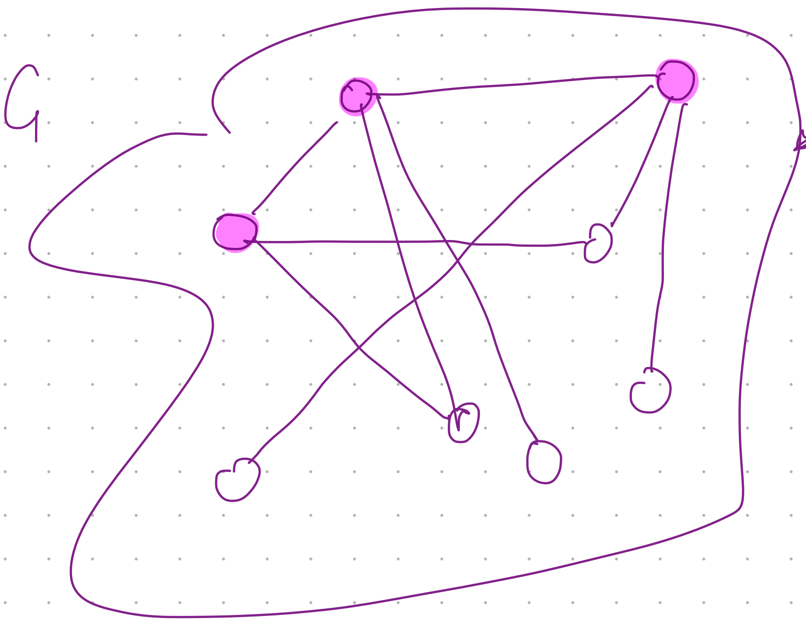
INDSET is NP-Hard

INDSET $\in$ NP.

$\Longrightarrow$ INDSET $\in$ NP-Complete.

VERTEXCOVER : Given a graph G and a size k, does G have a vertex cover of cardinality $\leq k$?

G



$C \subseteq V$ is a vertex cover if $\forall (u,v) \in E$

$u \in C$ or $v \in C$.

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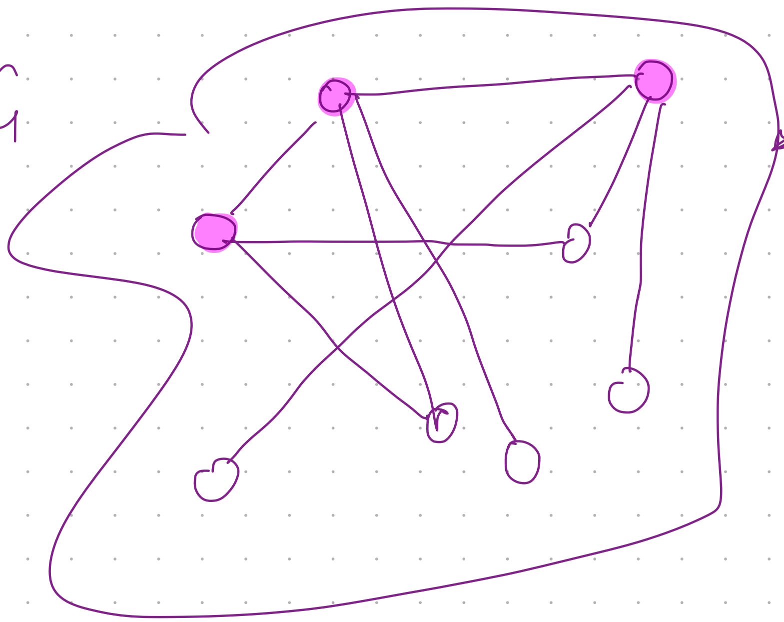

$C \subseteq V$ is a vertex cover if $\forall (u,v) \in E$

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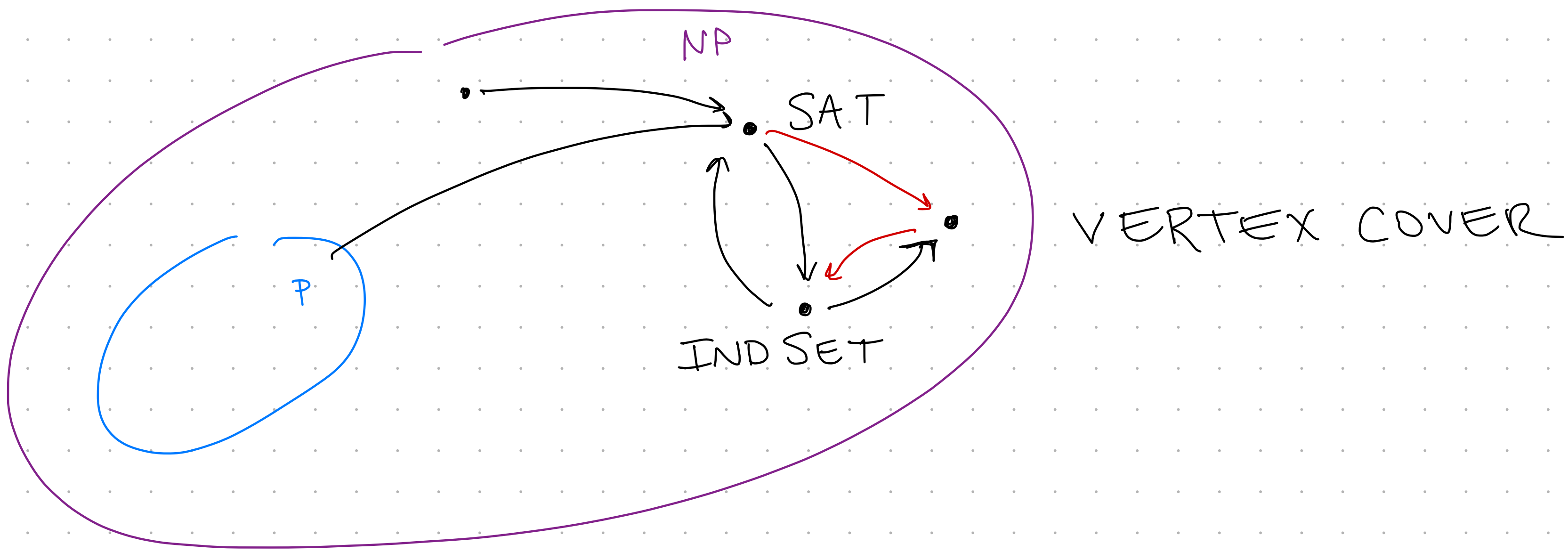
<u>Theorem</u> VERTEX COVER is NP-Hard.



Theorem VERTEX COVER is NP-Hard.

By Reduction from INDSET.

VERTEX COVER:

$C \subseteq V$ is a vertex cover if for all

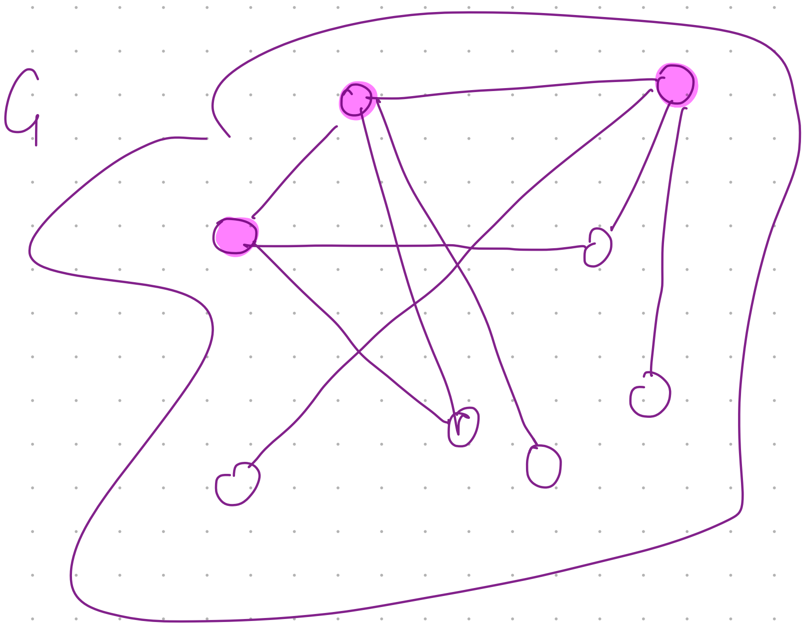
$(u,v) \in E \Rightarrow u \in C$ or $v \in C$.

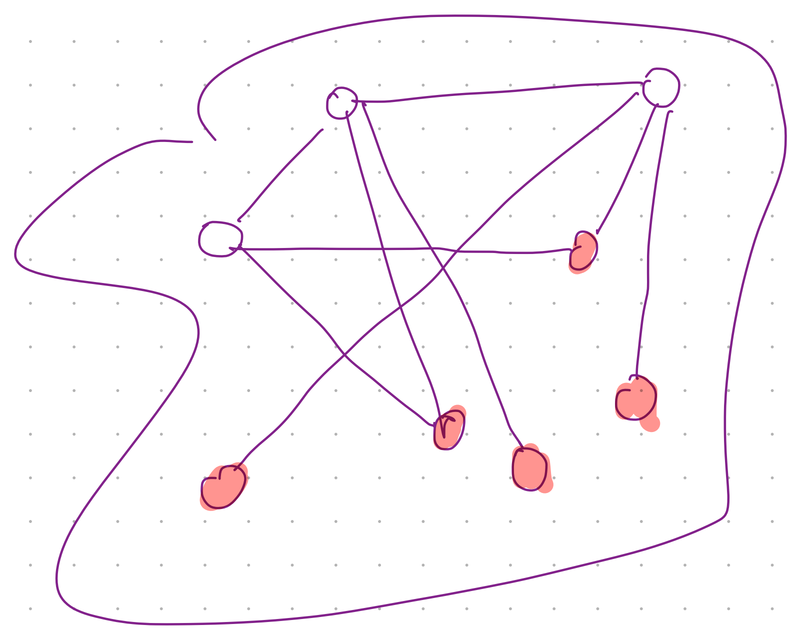
INDEPENDENT SET:

$S \subseteq V$ is an independent set if for all

$u \in S$ or $v \in S \Rightarrow (u,v) \notin E$

G





VERTEX COVER:

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INDEPENDENT SET:

$S \subseteq V$ is an independent set if for all

$u \in S$ or $v \in S \Rightarrow (u,v) \notin E$

Theorem. $C \subseteq V$ is a Vertex Cover

iff

$S = V \setminus C$ is an Independent Set

VERTEX COVER:

$C \subseteq V$ is a vertex cover if for all

$$(u,v) \in E \implies u \in C \text{ or } v \in C.$$

INDEPENDENT SET:

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$$u \in S \text{ or } v \in S \implies (u,v) \notin E$$

$IS \leq_p VC$

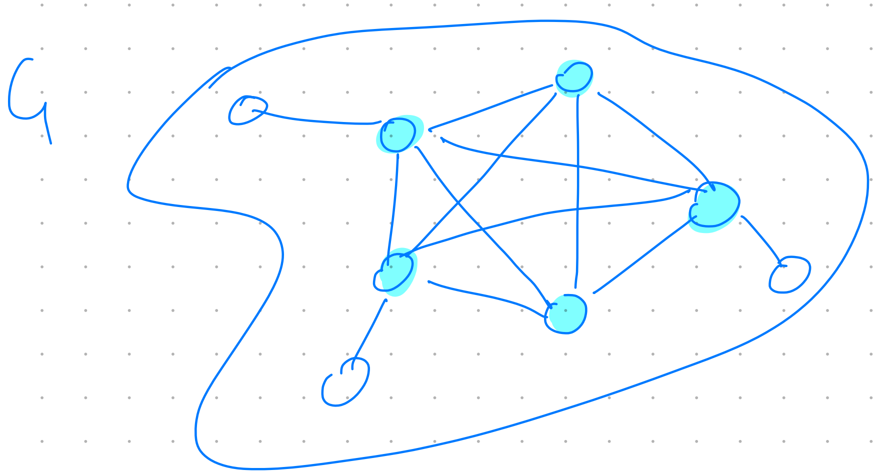
$$\boxed{\text{Given } \langle G, k \rangle, \text{ return } \langle G, n-k \rangle}$$

poly-time reduction

$$\langle G, k \rangle \in IS \iff \langle G, n-k \rangle \in VC.$$

CLIQUE: Given a graph $G = (V, E)$ does there exist a set of $\geq k$ mutually-connected vertices?

$K \subseteq V$ s.t. $\forall u, v \in K$, $(u,v) \in E$.



5-clique

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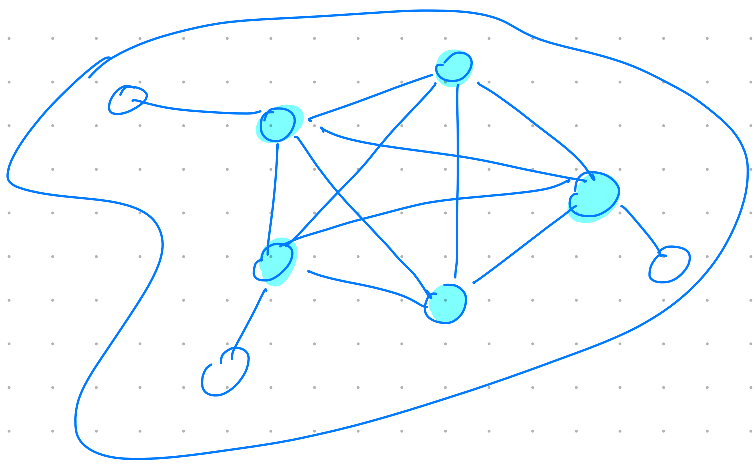
G



5-clique

Theorem. CLIQUE is NP-Hard.

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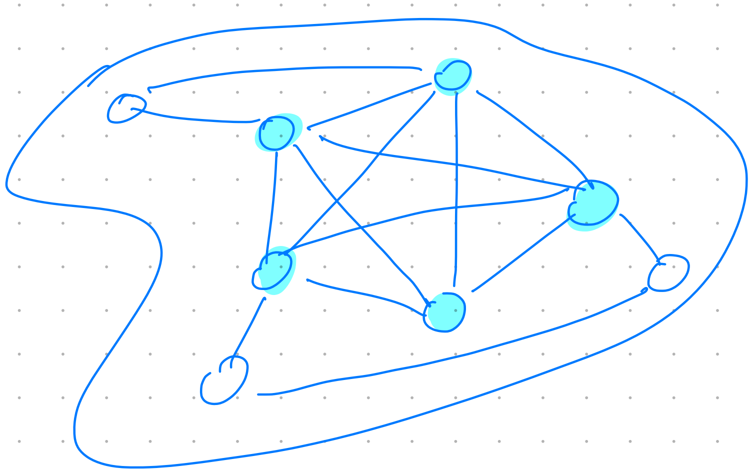
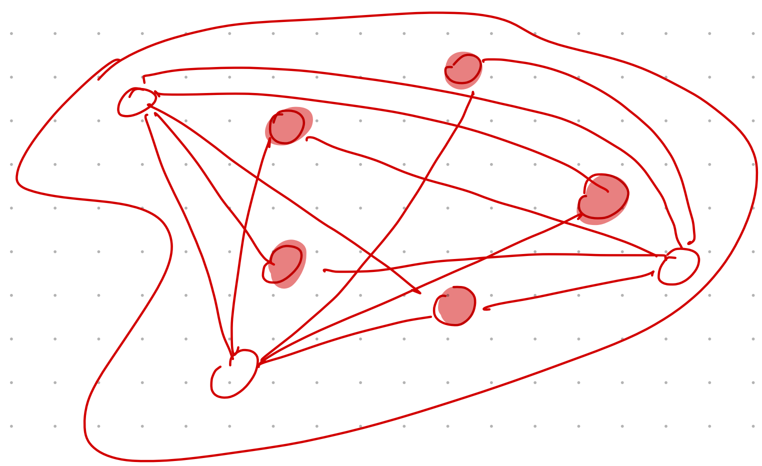
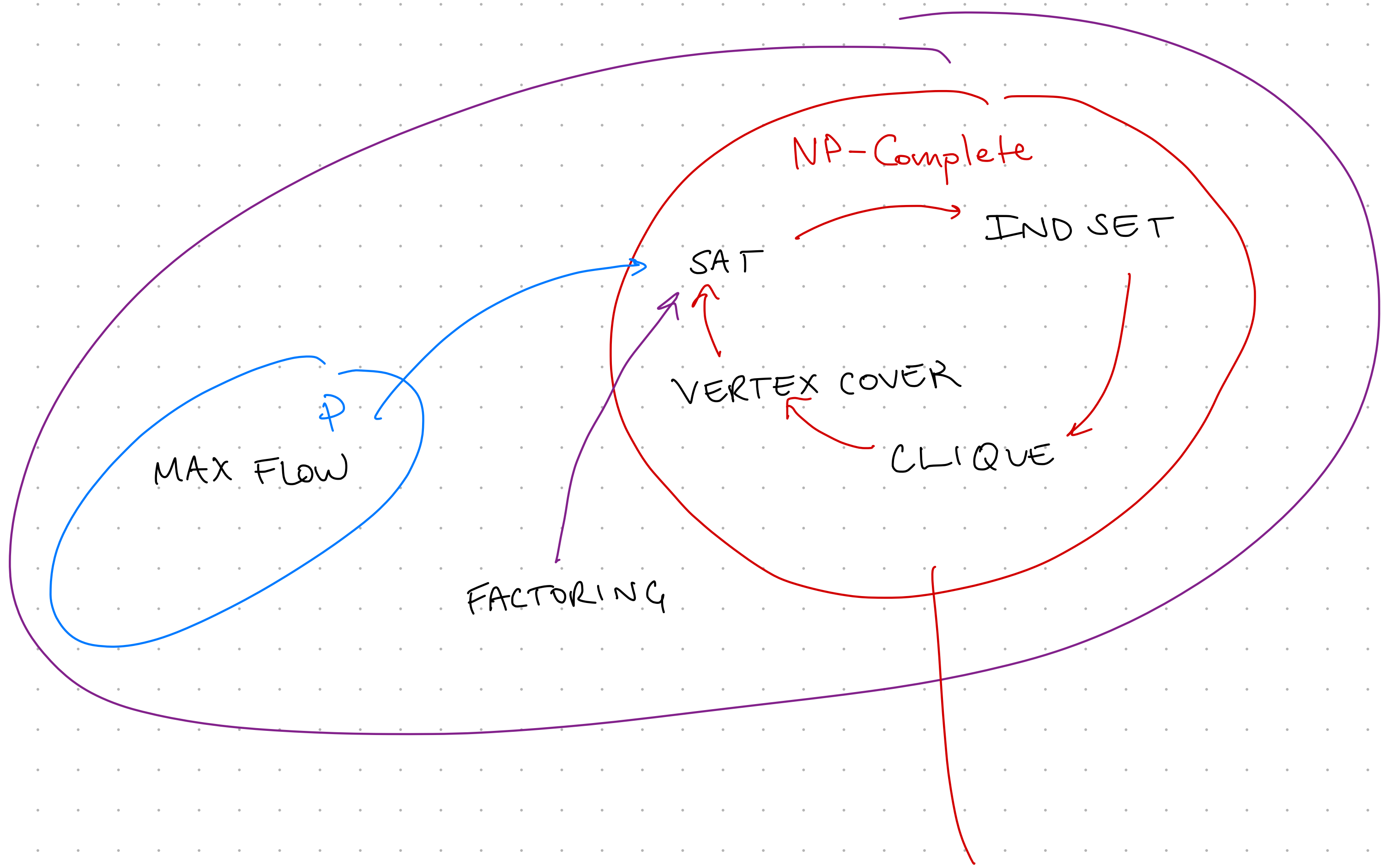
Theorem. G has a k-Clique iff $G' = (V, V \times V \setminus E)$ has a k-IS.

↳ Complement edge set.

G


G'




NP-Complete problems form an <u>equivalence class</u> under poly-time reductions

Covering & Packing Problems

SET COVER: Given a finite universe $\mathcal{U} = \{u_1, \dots, u_m\}$ and a collection of sets $S_1, \dots, S_n \subseteq \mathcal{U}$

Does there exist a collection of at most k sets s.t. $S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k} = \mathcal{U}$?

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<u>Theorem</u>. SET COVER is NP-Hard.

<u>Pf</u>. VC $\leq_p$ SET COVER.

$$\mathcal{U} = E \qquad S_v = \left\{ e \in E : e = (u,v) \text{ for } u \in V \right\}$$

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Does there exist a collection of at least k sets s.t. $S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k} = \emptyset$?

Theorem. SET PACKING is NP-Hard.

Pf. $IS \leq_p$ SET COVER.

$\mathcal{U} = V \qquad S_e = \{u, v : (u,v) = e \in E\}$