28 February 2025

Max Flow / Min Cut & König's Theorem

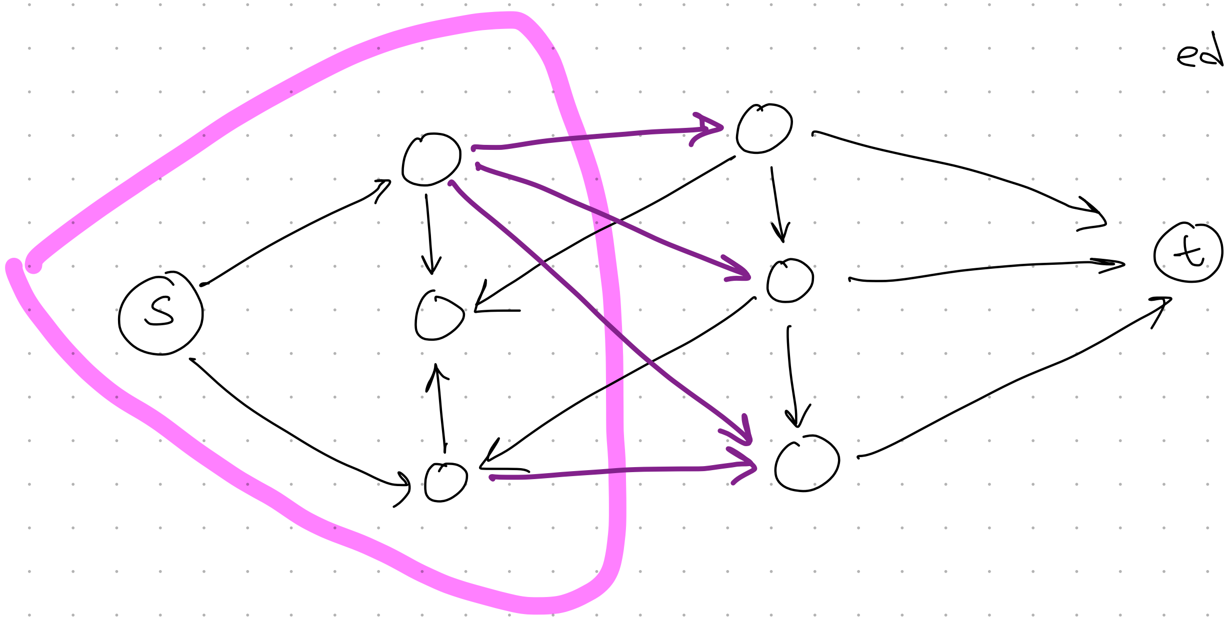
Plan

* Complete proof of Max Flow / Min Cut
* Announcements
* König's Theorem.

Consider an st-Cut of G. $S \subseteq V$ $T = V \setminus S$

s.t. $s \in S$ $t \in T$.

Capacity of an st-cut. $Cap(S,T) = \underbrace{\sum_{\substack{uv \in E \\ u \in S, v \in T}}}_{\text{edges from } S \text{ to } T.} c_{uv}$



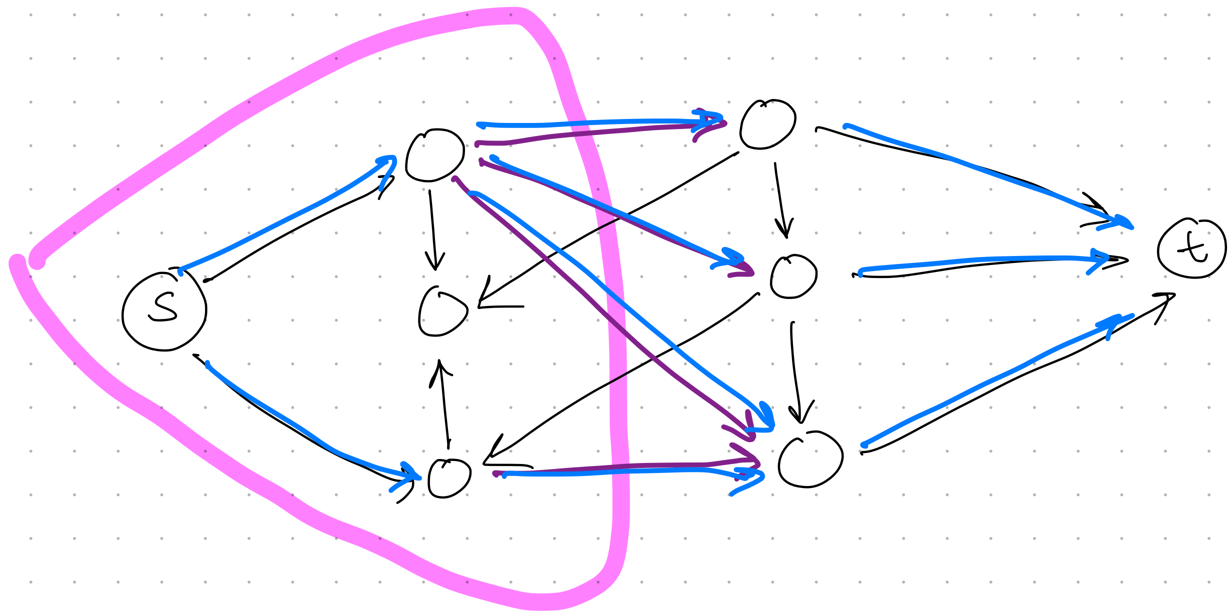
Natural upper bound on FlowOut(s).

$val(f) \leq Cap(S,T)$

Theorem (Max Flow / Min Cut)

For every flow network G, s, t, $c: E \to \mathbb{R}^+$

$$\boxed{\max_{\text{flows } f} \text{val}(f) = \min_{\substack{\text{st-cut}\\ S,T}} \text{Cap}(S,T)}$$



Finding a maximum flow is equivalent to finding a minimum st-cut.

Proof by Analysis of Ford-Fulkerson.

Lemma. For any flow f and any st-Cut $S \subseteq V$.

$$val(f) = FlowOut^f(S) - FlowIn^f(S)$$

Claim. When FF terminates w/ f, there exists st-Cut S, T such that

$$val(f) = FlowOut^f(S) = Cap(S, T)$$

Lemma. For any st-cut $S \subseteq V$, and any flow f.

$$\mathrm{val}(f) = \mathrm{FlowOut}^f(S) - \mathrm{FlowIn}^f(S)$$

$$= \sum_{u \in S, v \in T} f_{uv} - \sum_{v \in T, u \in S} f_{vu}$$

Intuition. Flows between $u, u' \in S$ "cancel"

Similar for $v, v' \in T$.

Net flow crosses cut.

Lemma. For any st-Cut $S \subseteq V$ and any flow f.

$$\mathrm{val}(f) = \sum_{u \in S, v \in T} f_{uv} - \sum_{v \in T, u \in S} f_{vu}$$

Pf. $\mathrm{val}(f) = \mathrm{FlowOut}(s)$

$$= \mathrm{FlowOut}(s) + \sum_{u \in S \setminus \{s\}} \underbrace{\left(\mathrm{FlowOut}(u) - \mathrm{FlowIn}(u) \right)}_{0}$$

$0 \Leftarrow$ CONSERVATION

Lemma. For any st-cut $S \subseteq V$ and any flow f.

$$\mathrm{val}(f) = \sum_{u\in S, v\in T} f_{uv} - \sum_{v\in T, u\in S} f_{vu}$$

Pf.

$$\begin{aligned}
\mathrm{val}(f) &= \mathrm{FlowOut}(s) \\
&= \mathrm{FlowOut}(s) + \sum_{u\in S\setminus\{s\}} \underbrace{\left(\mathrm{FlowOut}(u) - \mathrm{FlowIn}(u)\right)}_{0 \,\leftarrow\, \text{CONSERVATION}} \\
&= \sum_{u\in S}\left(\sum_{u'\in S} f_{uu'} + \sum_{v\in T} f_{uv} - \sum_{u'\in S} f_{u'u} - \sum_{v\in T} f_{vu} \right) \\
&= \underbrace{\sum_{u,u'\in S} \left(f_{uu'} - f_{uu'}\right)}_{0} + \sum_{u\in S, v\in T} f_{uv} - \sum_{u\in S, v\in T} f_{vu}
\end{aligned}$$

Flow that circulates w/in S

Claim. When FF terminates w/ f, there exists st-cut S, T such that

$$\mathrm{val}(f) = \mathrm{FlowOut}^f(S) = \mathrm{Cap}(S, T)$$

Ford - Fulkerson Algorithm

Initialize $f_e = 0 \quad \forall e \in G$.

Repeat.

* Compute G^f the residual of current flow.

* if there exists an st-path in G^f
  - Push flow along path
  - Update flow f.

* if no st-path in G^f
  - Return f

To show: f witnesses a min cut.

Claim. When FF terminates w/ f, there exists st-Cut S, T such that

$$\text{val}(f) = \text{FlowOut}^f(S) = \text{Cap}(S,T)$$

Termination Condition: No st-path in G^f.

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$

$T = V \setminus S$.

Note: $t \in T$ by termination condition.

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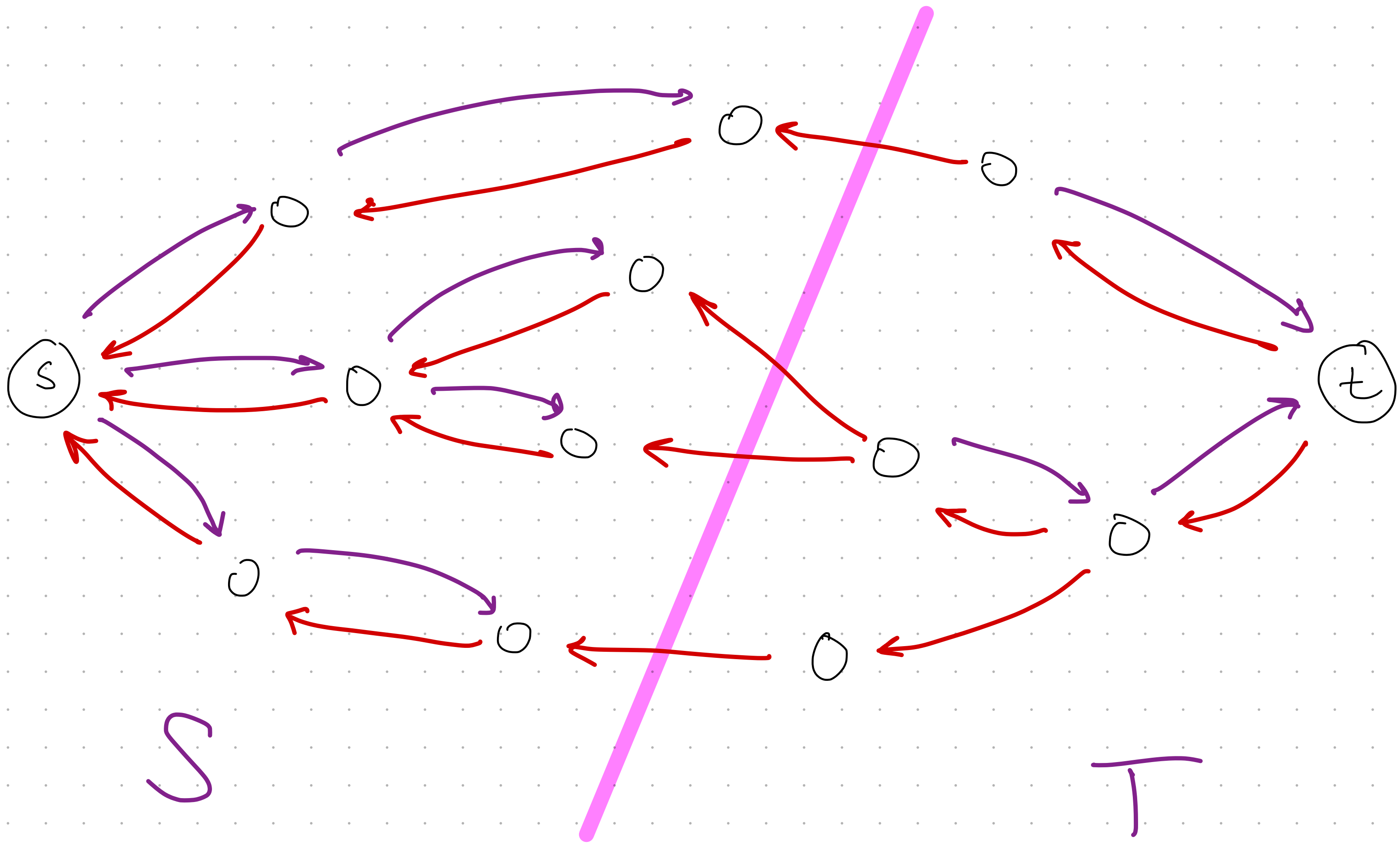


Claim. When FF terminates w/ f, there exists st-Cut S, T such that

$$val(f) = FlowOut^f(S) = Cap(S,T)$$

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$

We show:

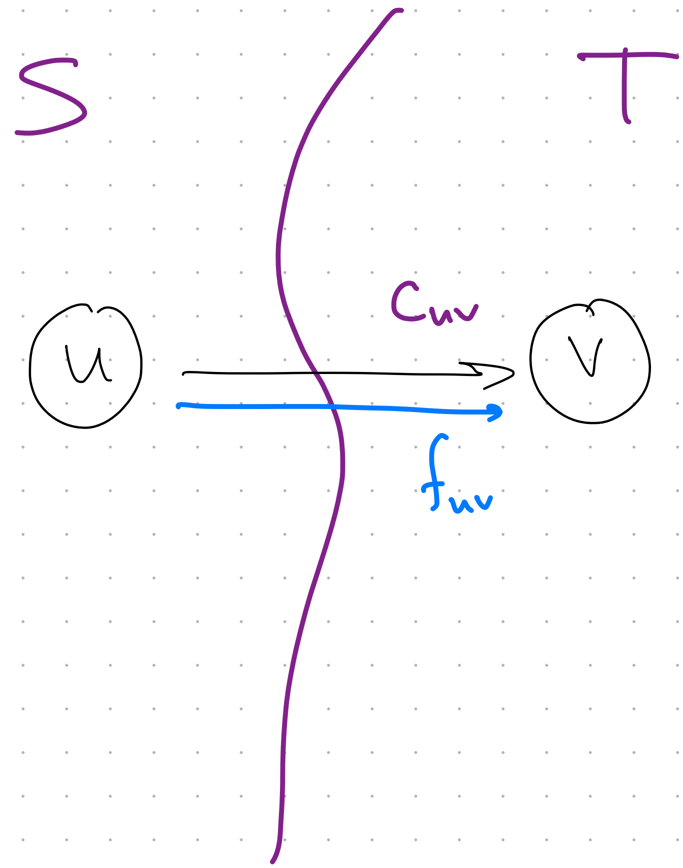
$\forall u \in S, v \in T$: $f_{uv} = c_{uv}$.

$\forall v \in T, u \in S$: $f_{vu} = \emptyset$.

$\Downarrow$

establishes the Claim above.

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$



Suppose $uv \in E$.

<u>Claim</u>. $f_{uv} = c_{uv}$.

<u>Pf</u>. Suppose $f_{uv} < c_{uv}$.

$\Longrightarrow c^f(uv) = c_{uv} - f_{uv} > 0$

contradicting v not reachable from u.

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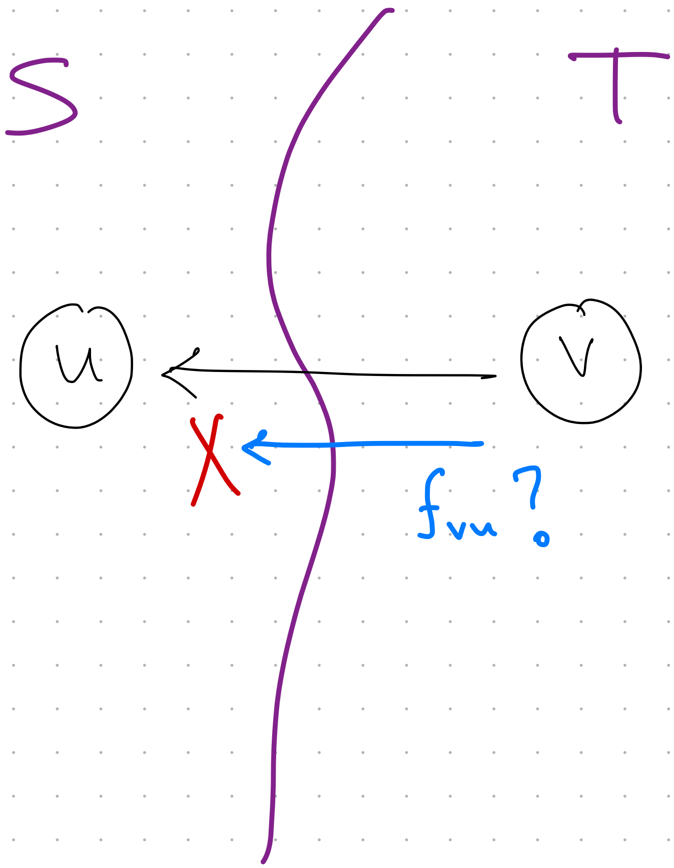


Suppose $vu \in E$.

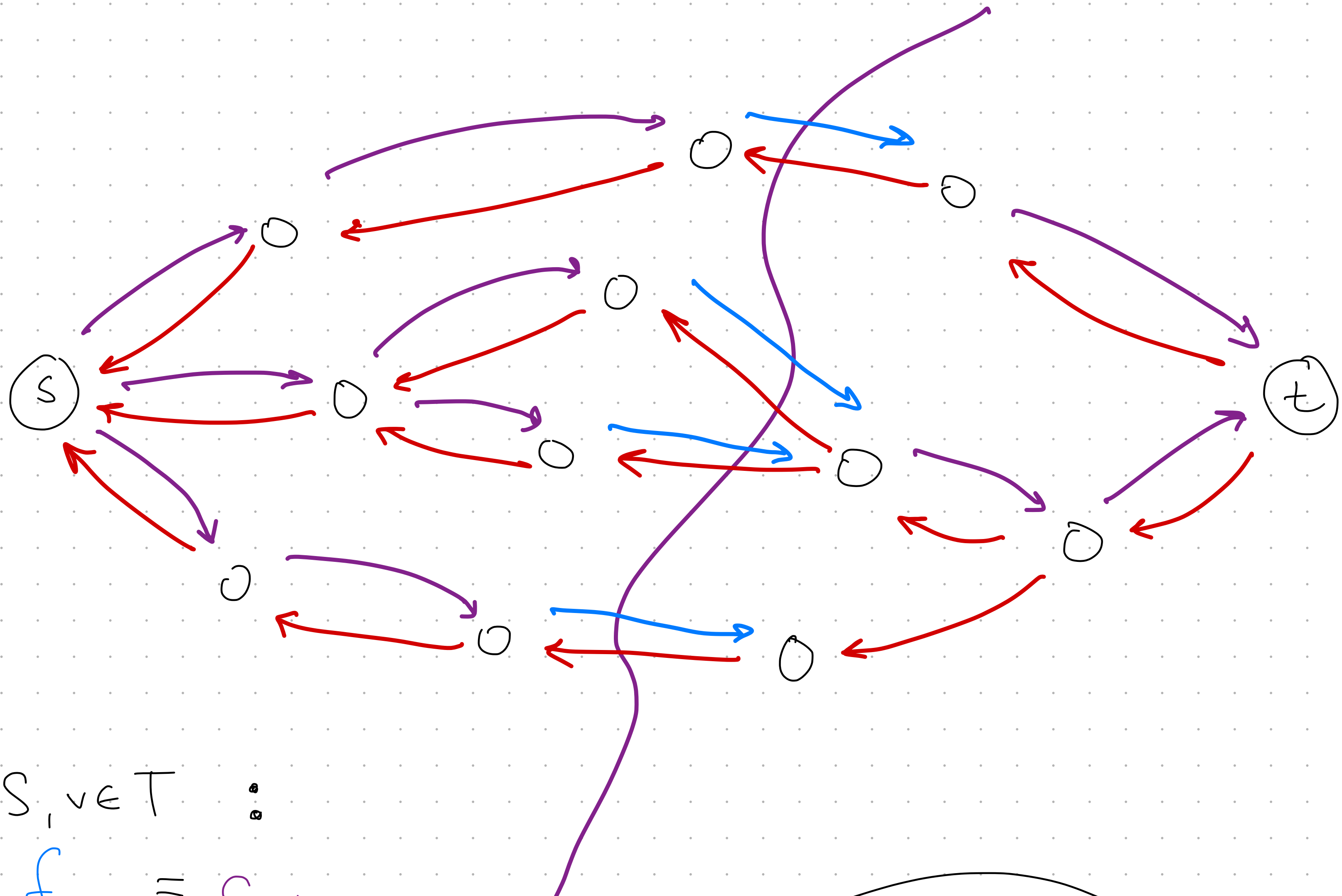
<u>Claim</u>. $f_{vu} = 0$.

<u>Pf</u>. Suppose $f_{vu} > 0$

$\Longrightarrow c^f(uv) = f_{vu} > 0$

contradicting v not reachable from u.

Termination Condition: <u>No st-path in G^f.</u>



$\forall u \in S, v \in T$:

$$f_{uv} = c_{uv}.$$

$\forall v \in T, u \in S$:

$$f_{vu} = \emptyset.$$

$$val(f) = \sum_{u\in S, v\in T} f_{uv} - \underbrace{\sum_{v\in T, u\in S} f_{vs}}_{\emptyset} = \sum_{u\in S, v\in T} c_{uv} = Cap(S,T)$$

Summary:

* For any flow f and any st-cut S, T

$$val(f) \leq Cap(S, T)$$

Summary:

* For any flow f and any st-cut S,T

$$\mathrm{val}(f) \leq \mathrm{Cap}(S,T)$$

* Ford-Fulkerson terminates w/ a flow f^* s.t.
$\exists$ st-cut S^*, T^*

$$\mathrm{val}(f^*) = \mathrm{Cap}(S^*, T^*)$$

Summary:

* For any flow f and any st-cut S, T

$$\mathrm{val}(f) \leq \mathrm{Cap}(S, T)$$

* Ford-Fulkerson terminates w/ a flow f^* s.t. $\exists$ st-cut S^*, T^*

$$\mathrm{val}(f^*) = \mathrm{Cap}(S^*, T^*)$$

* Implies both:
  - $\Longrightarrow$ Ford-Fulkerson is correct.
  - $\Longrightarrow$ Max-Flow = Min-Cut.

Summary:

* Ford - Fulkerson terminates in $\leq val(f^*)$ iterations

$$\Rightarrow RT: O(\underbrace{val(f^*)} \cdot m)$$

"pseudopolynomial"

$\Rightarrow$ To be polynomial we need that capacities are bounded polynomially.

Edmonds-Karp / Dinic

$\Downarrow$

implementation of FF that runs in polynomial time

Announcements.

* HW 3 ongoing
  - ↳ Ask on Ed for suggestions on programming problem
  - ↳ Emphasis is on reductions

Two directions.

($\Rightarrow$) if A has soln $\Rightarrow$ R(A) has soln.

($\Leftarrow$) if R(A) has soln $\Rightarrow$ A has soln

* Recitation on Saturday.
  - ↳ Practice Problems!

* HW2 Grades released shortly

* Prelim #1 Solutions released shortly.

König's Theorem. ← Key application of Max Flow / Min Cut.

Defn. Given an undirected graph $G = (V, E)$ a Vertex Cover $C \subseteq V$ is a collection of vertices s.t.

for every edge $(u,v) \in E$

$u \in C$ OR $v \in C$.

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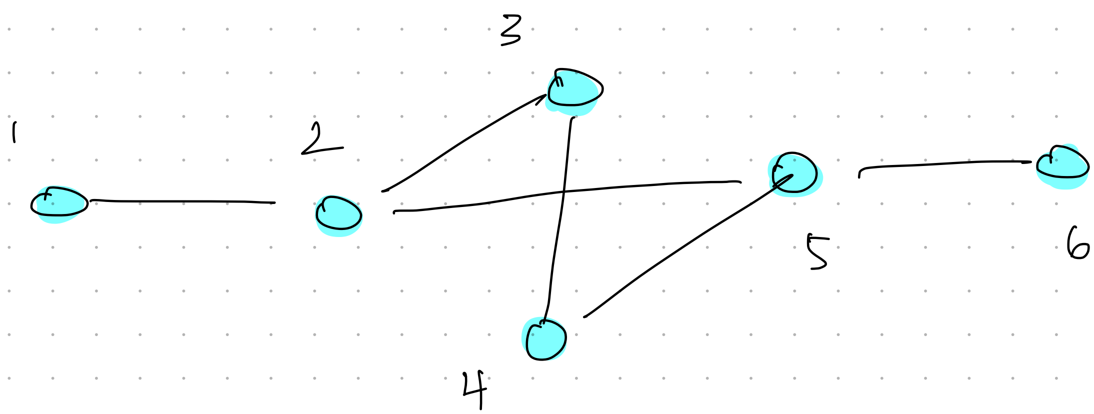

TRIVIAL VC

$C = V$

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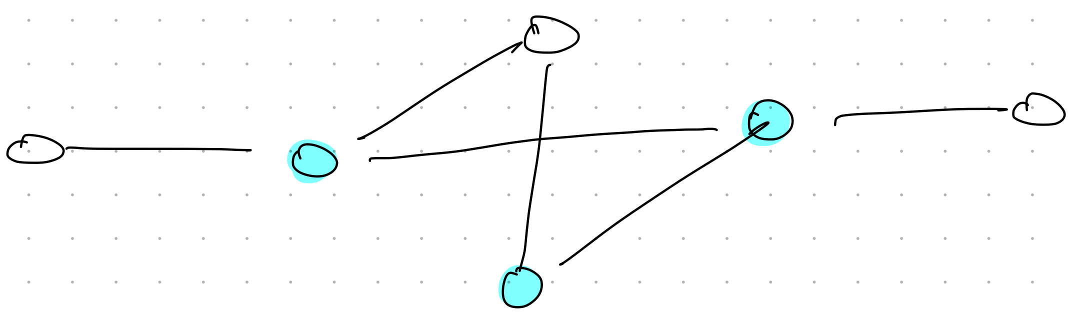


Non-trivial VC

↓

small C.

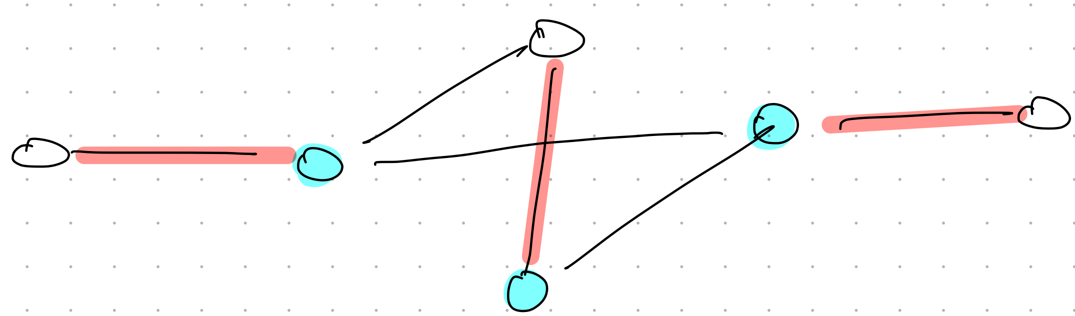
König's Theorem

Given a bipartite graph $G=(V,E)$.

The minimum cardinality of a vertex cover in G

equals

maximum cardinality of a matching in G.



König's Theorem. In bipartite graph G,

$$\min_{\text{VC}} |C| = \max_{\text{matching}} |M|$$

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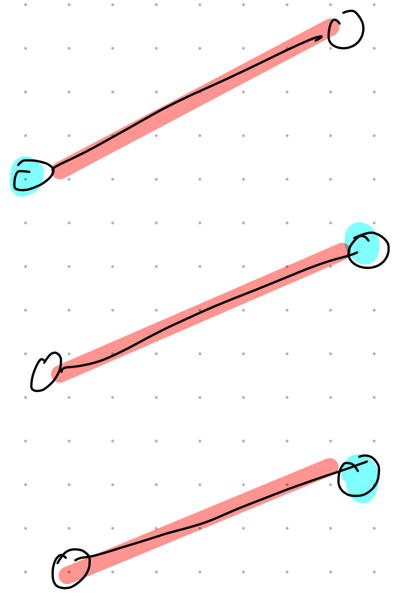
Easy Direction $\min_{VC} |C| \geq \max_{\text{matching}} |M|$.

Pf by picture



C must have at least one endpoint of every $e \in M$

König's Theorem. In bipartite graph G,

$$\underset{\text{VC}}{\min} |C| = \underset{\text{matching}}{\max} |M|$$

Easy Direction $\underset{\text{VC}}{\min} |C| \geq \underset{\text{matching}}{\max} |M|$.

Pf. Consider any vertex $v \in V$.
Adding v to C covers at most 1 edge from M (as it is a matching).

König's Theorem. In bipartite graph G,

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Pf. Consider any vertex $v \in V$.
Adding v to C covers at most
1 edge from M (as it is a matching).

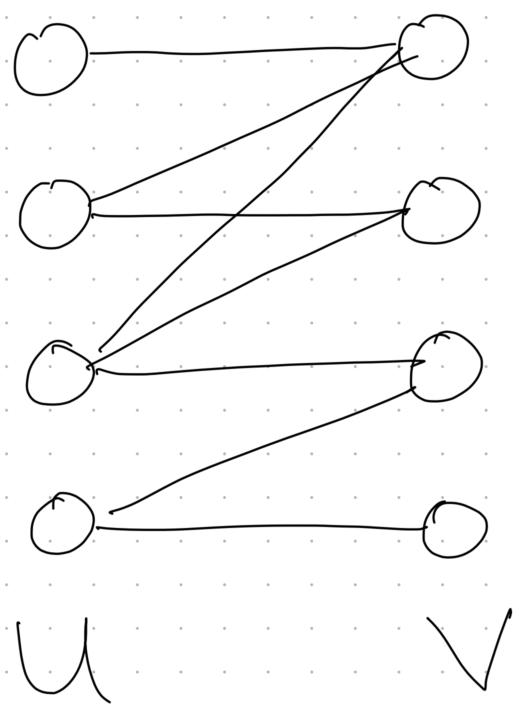
To cover every edge in E,
must cover every edge in M,
$\Longrightarrow$ Must use at least $|M|$ vertices.

König's Theorem. In bipartite graph G,

$$\underset{\text{VC}}{\min} |C| = \underset{\text{matching}}{\max} |M|$$

Hard Direction $\underset{\text{VC}}{\min} |C| \leq \underset{\text{matching}}{\max} |M|$

Reduce Matching to Flow



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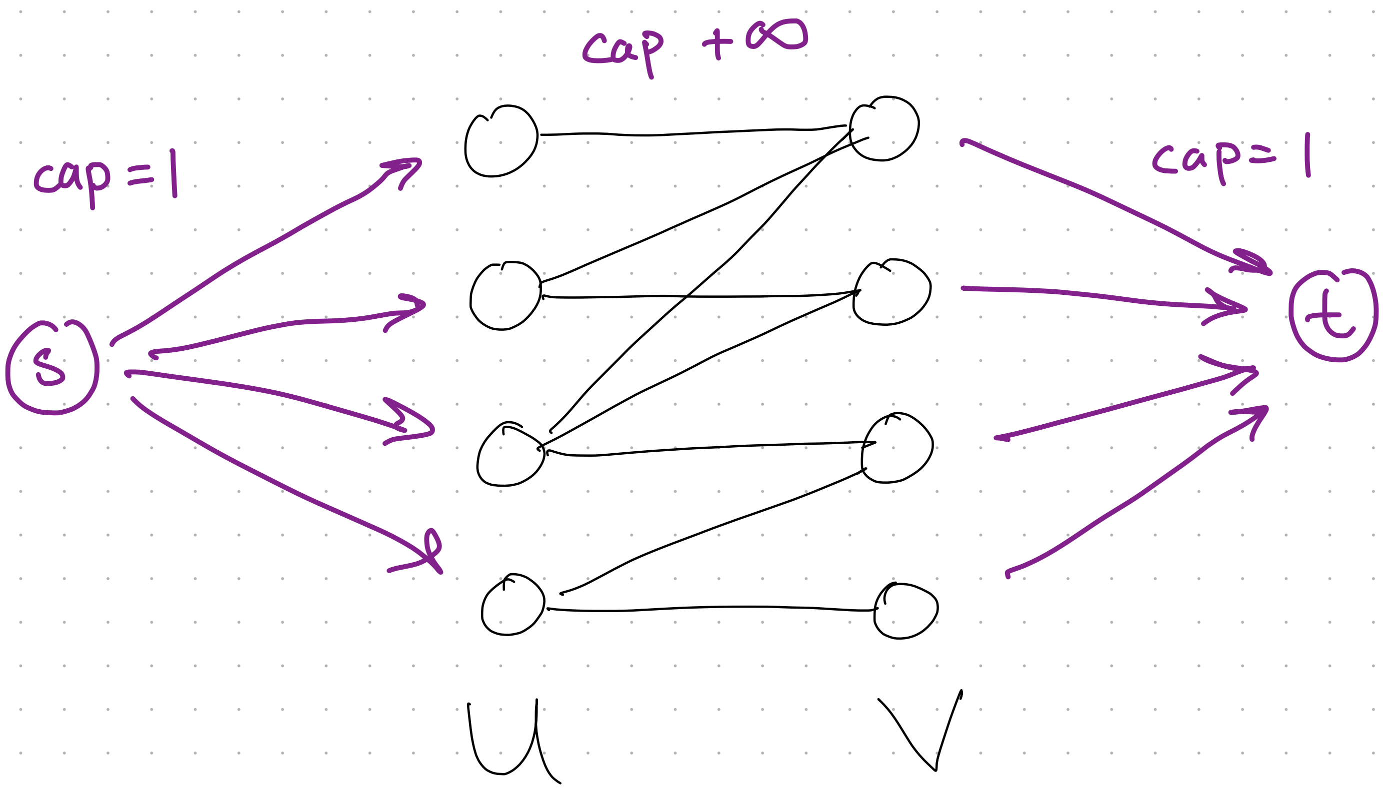
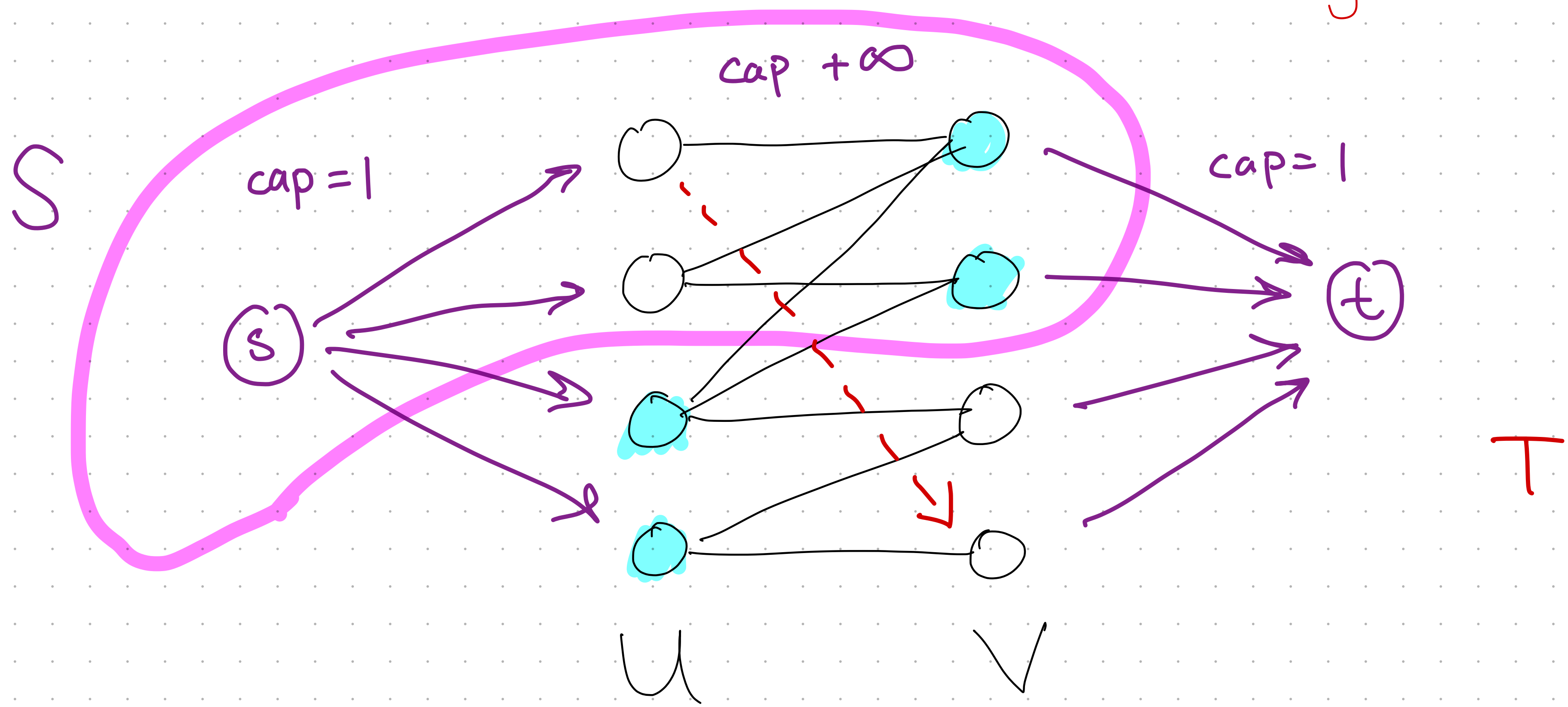
Reduce Matching to Flow



Hard Direction

min VC $|C| \le$ max matching $|M|$

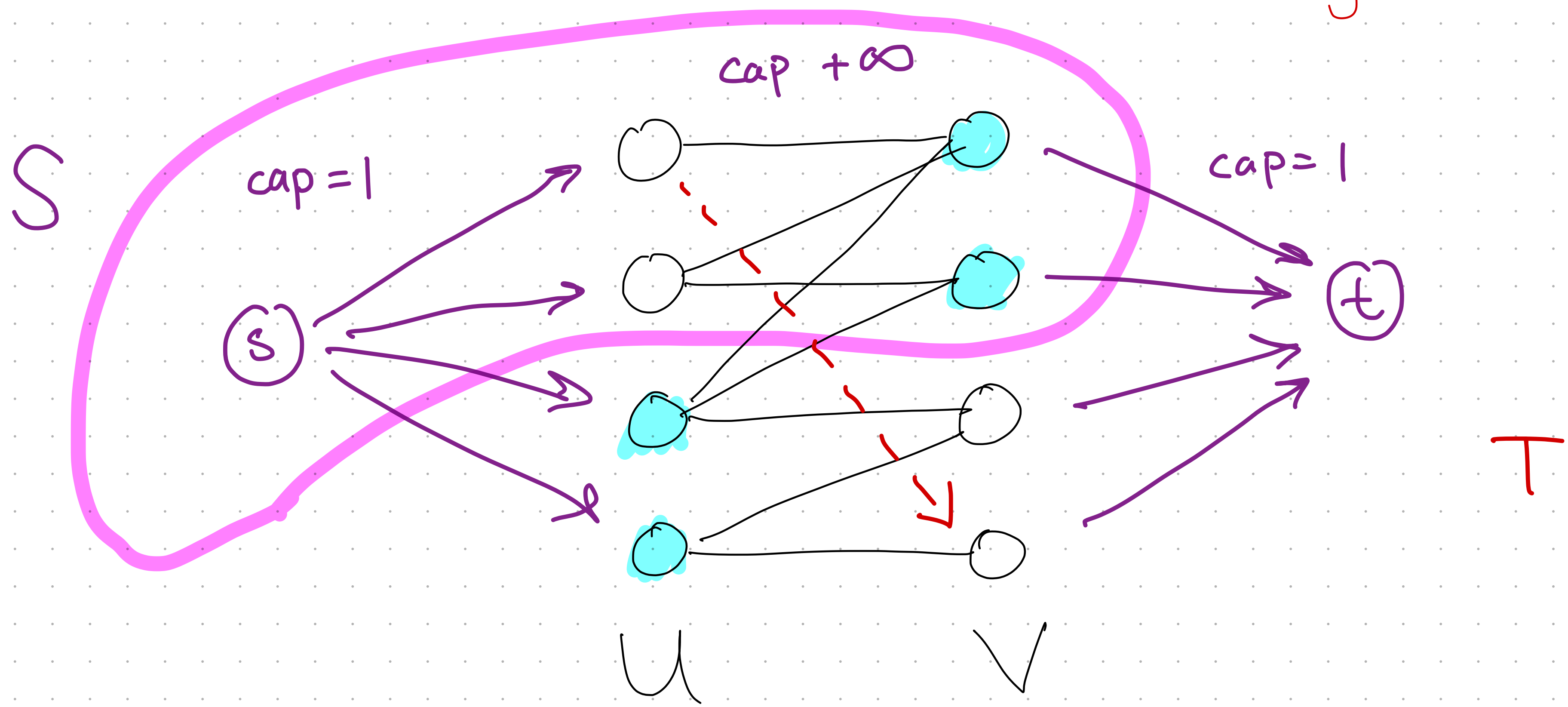


By $+\infty$ edge capacity

No $u \to v$ edges from $U \cap S \to V \cap T$

Hard Direction

min VC $|C| \leq$ max matching $|M|$



By $+\infty$ edge capacity

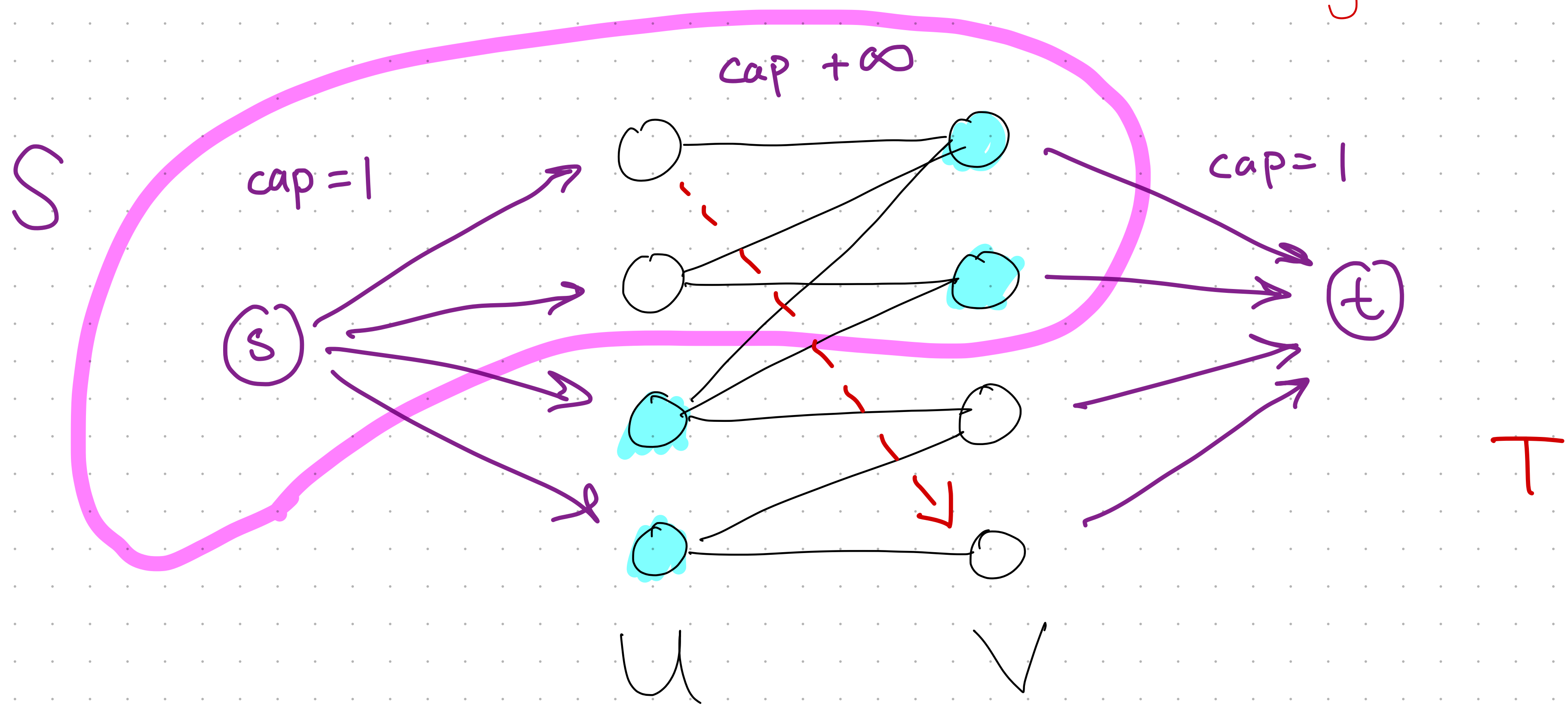
No $u \to v$ edges from $U \cap S \to V \cap T$

Consider $C = (U \cap T) \cup (V \cap S)$

C is VC: Every edge in G adjacent to $u \in U \cap T$ or $v \in V \cap S$

Hard Direction

min VC $|C| \le$ max matching $|M|$



By $+\infty$ edge capacity

No $u \to v$ edges from $U \cap S \to V \cap T$

Consider $C = (U \cap T) \cup (V \cap S)$

$|C| = Cap(S,T) = val(f) = |M|$ ∎

By earlier reduction.