26 February 2025 — More on Ford-Fulkerson

Plan

* Review of Ford-Fulkerson
* Execution of FF
* Announcements
* Analysis

Ford-Fulkerson Algorithm

Initialize $f_e = 0 \quad \forall e \in G$.

Repeat.

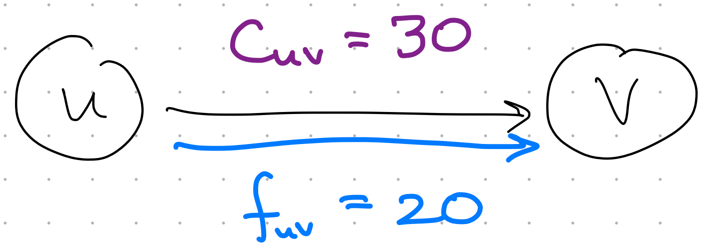
* Compute G^f the residual of current flow.
* if there exists an st-path in G^f
  - Push flow along path
  - Update flow f.

"Augmenting path"

* if no st-path in G^f
  - Return f

Residual Network

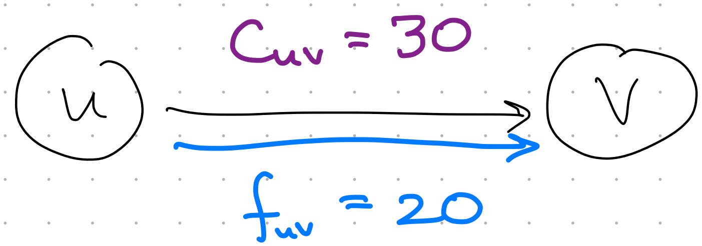
* For every edge in G.



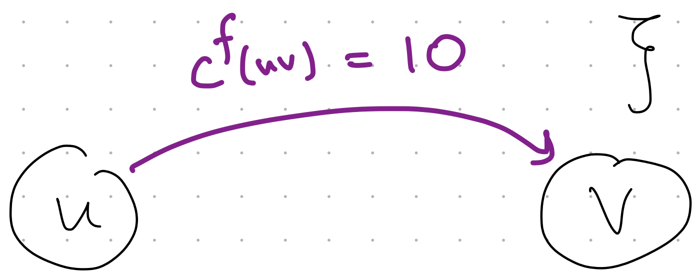
↳ two possible Residual edges in G^f

Residual Network

* For every edge in G.



↳ two possible Residual edges in G^f

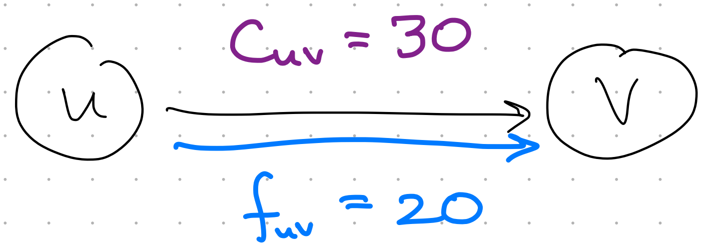


FORWARD RESIDUAL

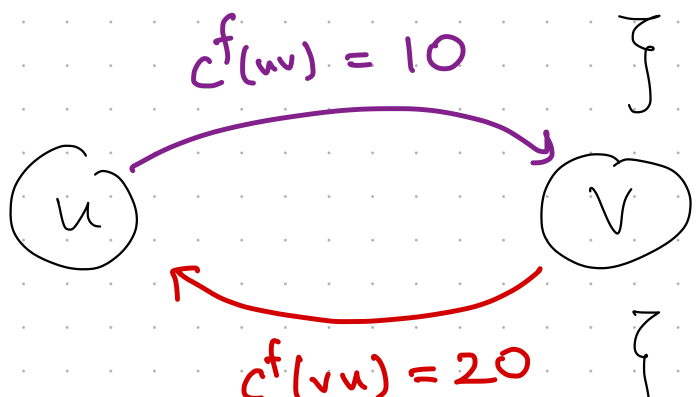
$$c^f(uv) = c_{uv} - f_{uv}$$

Residual Network

* For every edge in G.



↳ two possible Residual edges in G^f



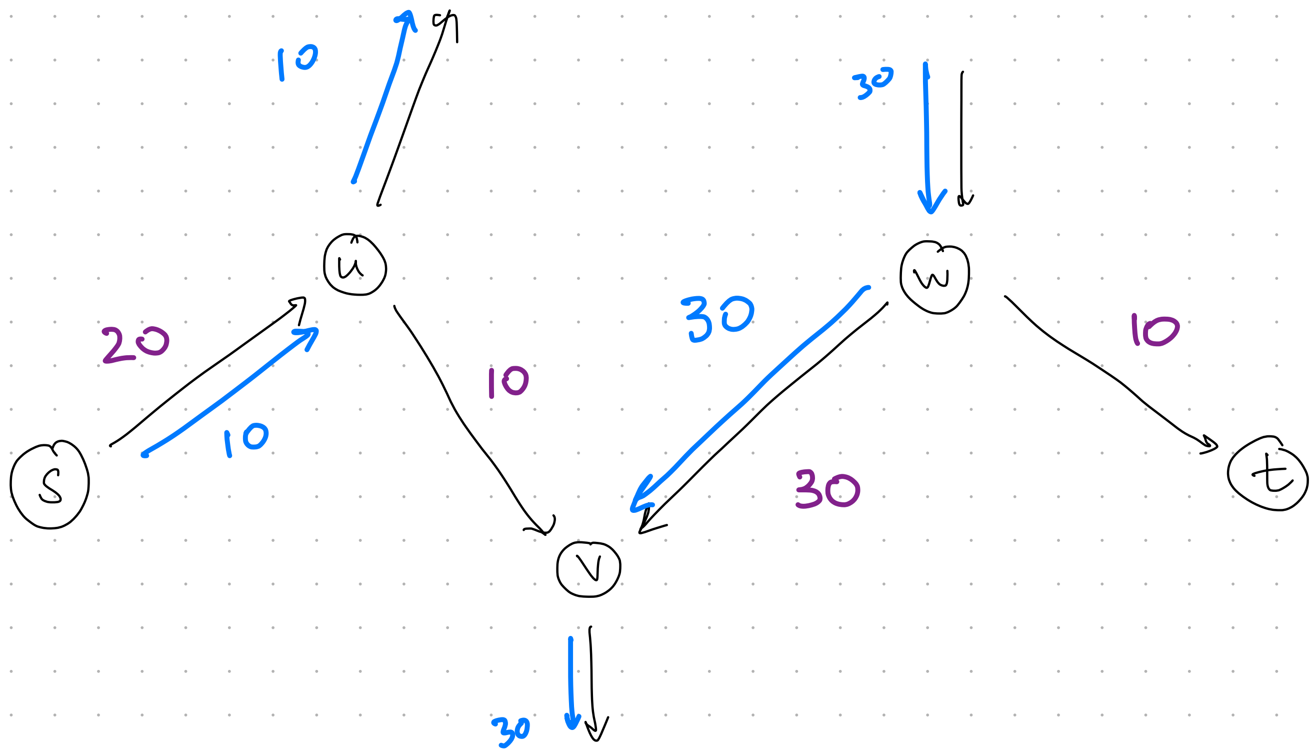
FORWARD RESIDUAL

$$c^f(uv) = c_{uv} - f_{uv}$$

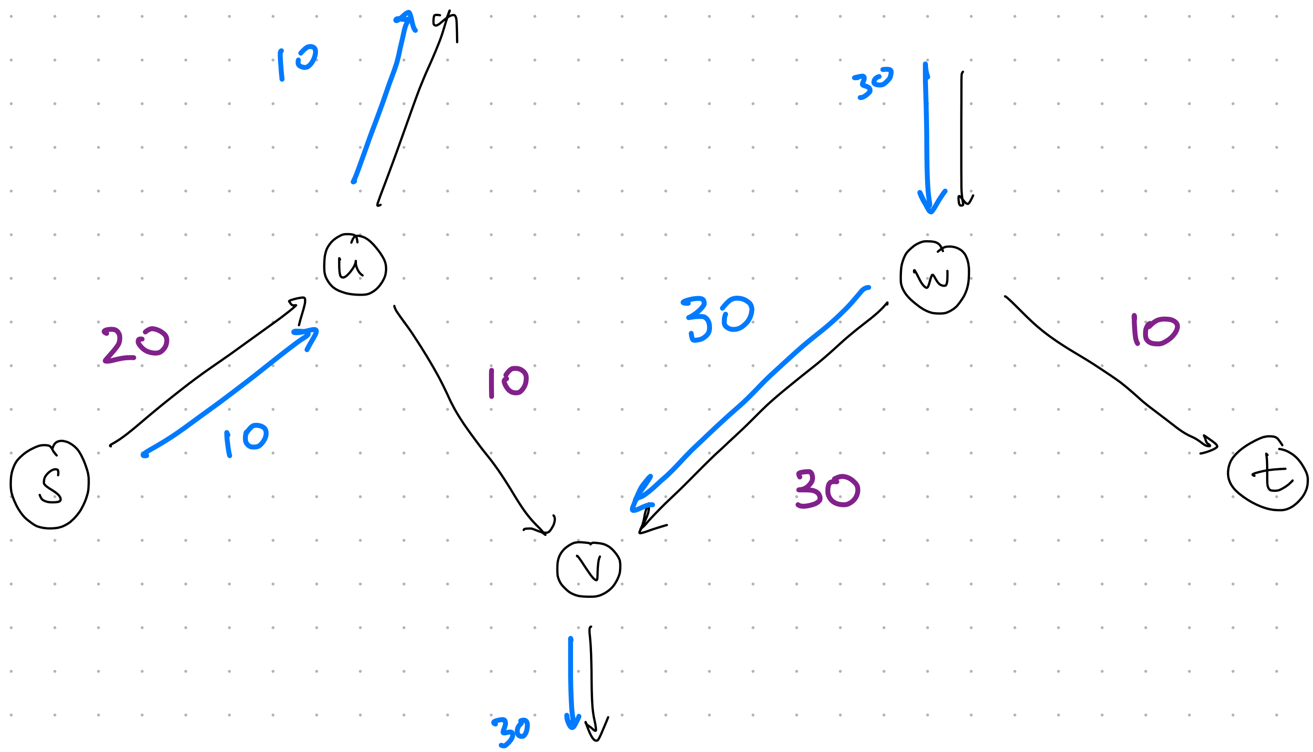
BACKWARD RESIDUAL

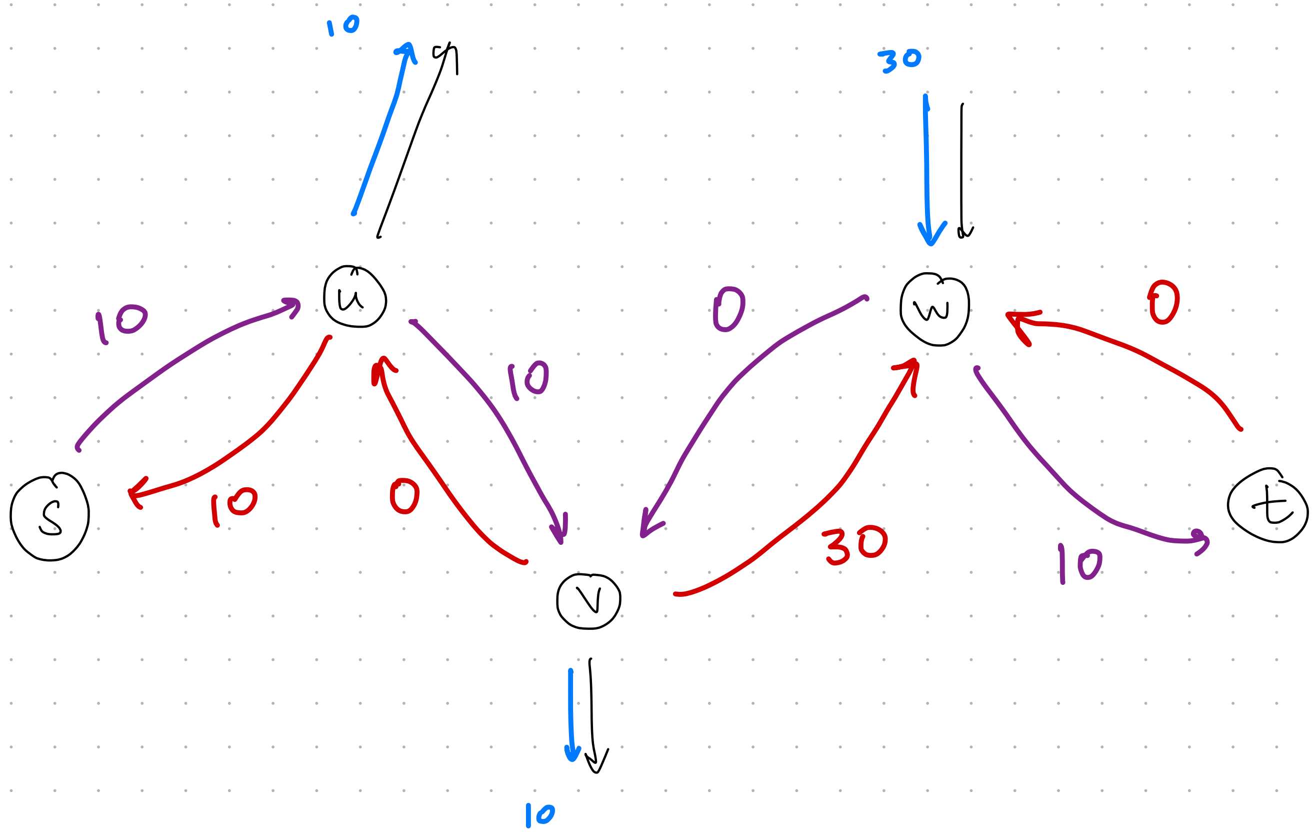
$$c^f(vu) = f_{uv}$$

Example. Augmenting path.

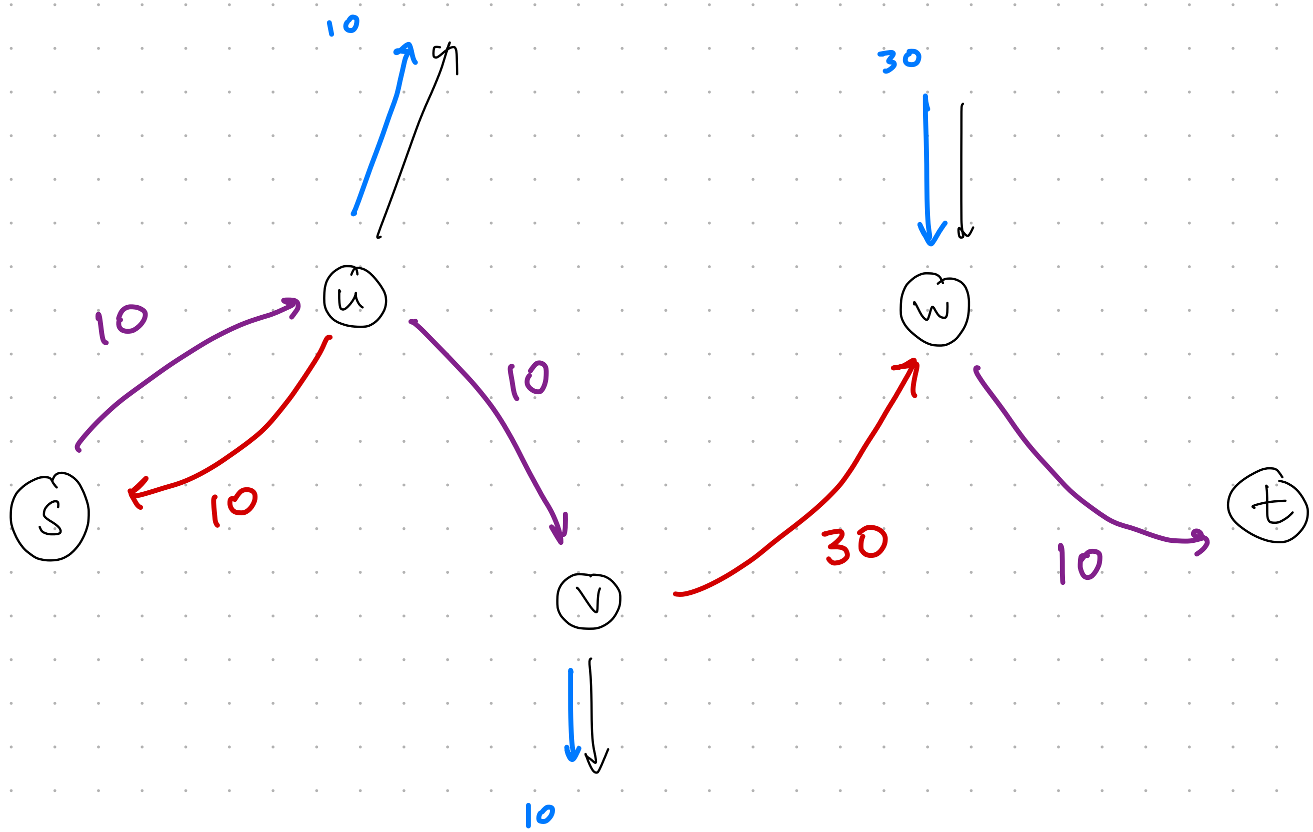


Example. Augmenting path.

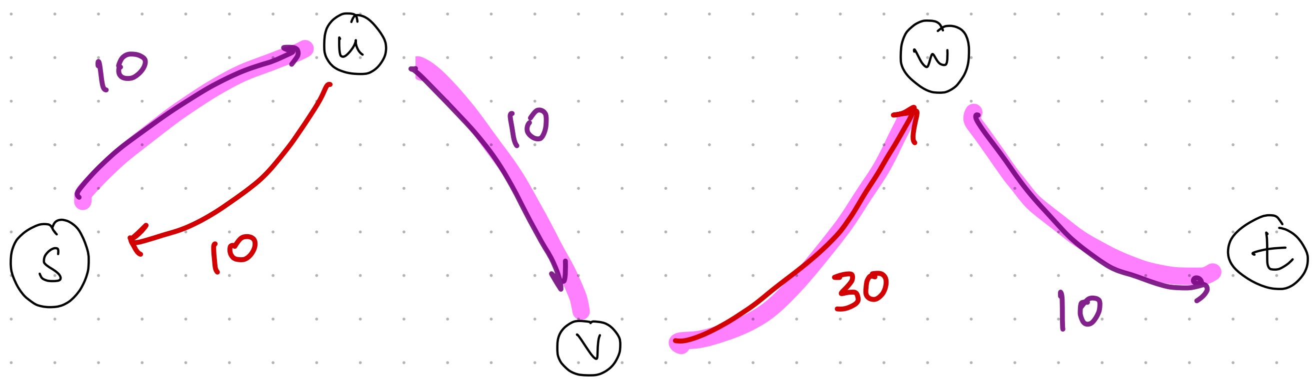




10
30
u
w
10
0
0
10
0
10
s
v
t
10
30
10
10


10
30
10
u
w
10
10
10
s
t
10
30
10
v
10

Augment the path w/ 10 units of flow.



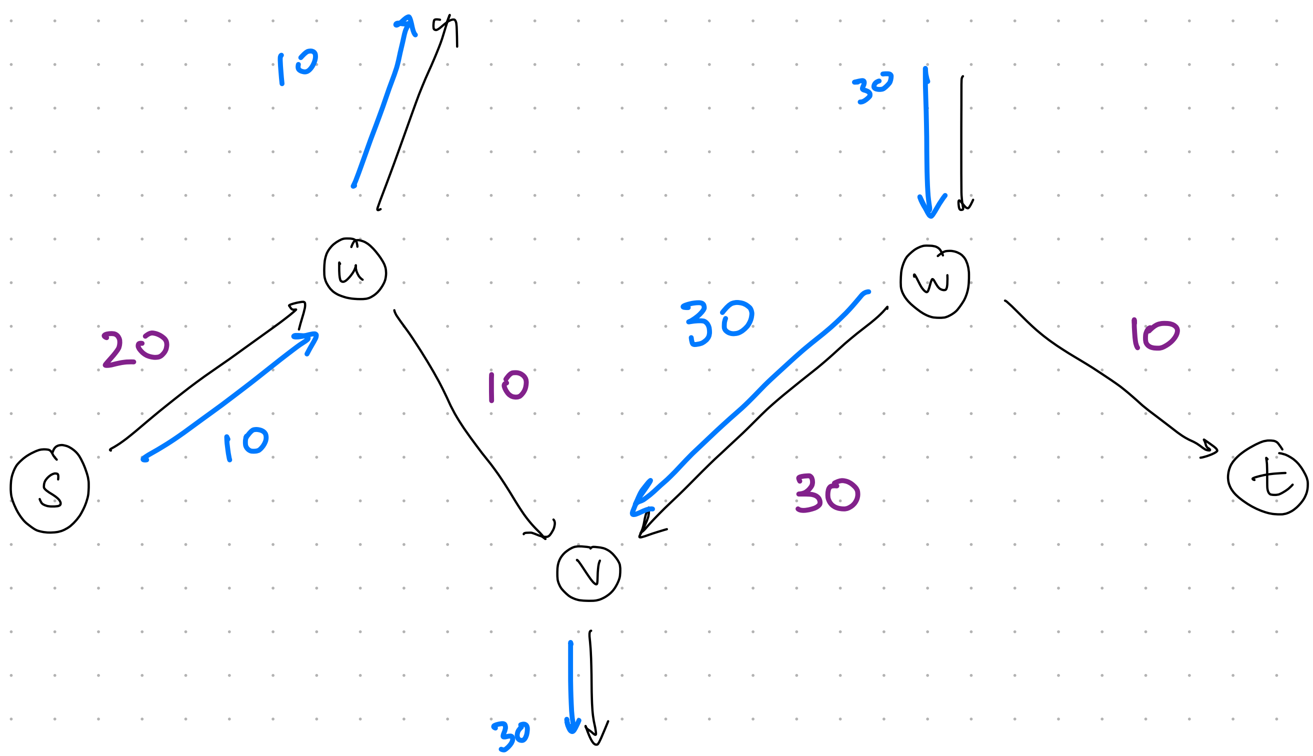
f_{su} $+10$

f_{uv} $+10$

f_{wv} -10

f_{wt} $+10$

Example. Augmenting path.

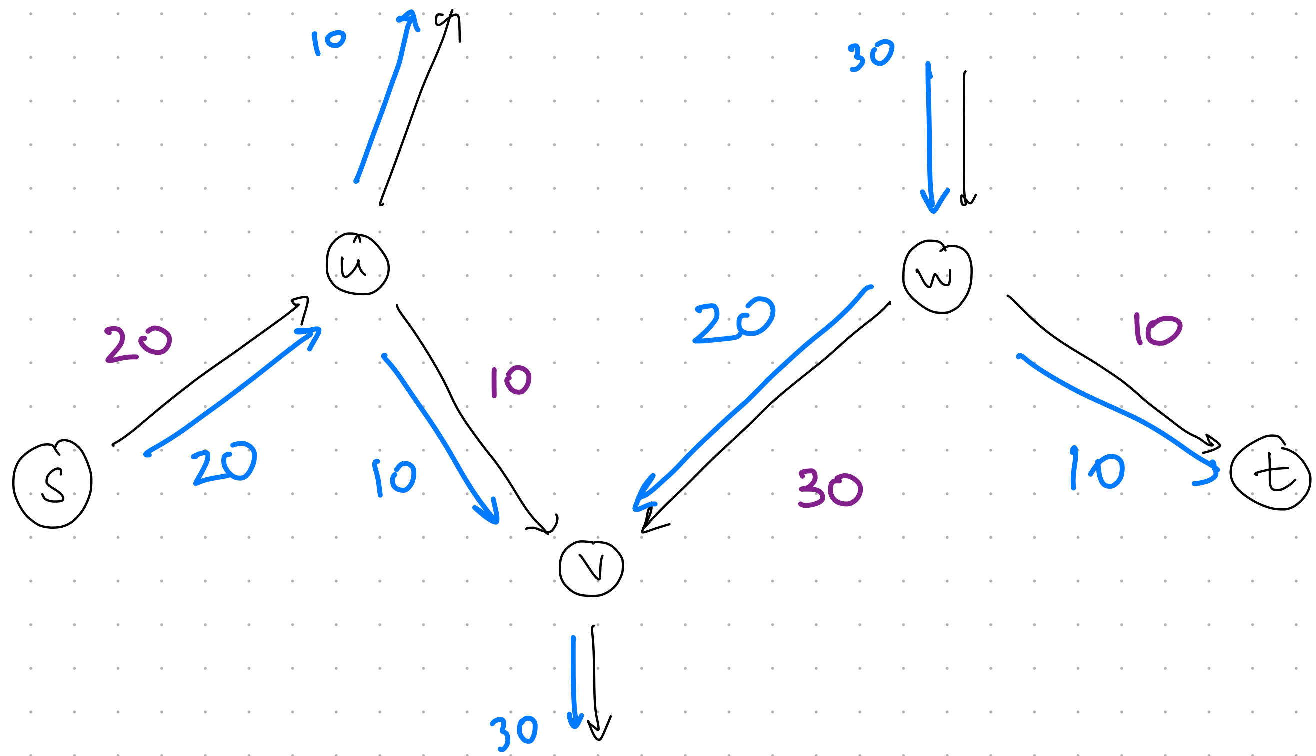


f_{su} $+10$

f_{uv} $+10$

f_{wv} -10

f_{wt} $+10$



Note.

* Flows outside of augmenting path unchanged.

* Augmenting path not necessarily a path in resulting flow

Announcements

* HW #3 Released Today
  - ↳ Involves programming problem.

Ford-Fulkerson Algorithm

Initialize $f_e = 0 \quad \forall e \in G.$

Repeat.

* Compute G^f the residual of current flow.
* if there exists an st-path in G^f
  - Push flow along path
  - Update flow f.
* if no st-path in G^f
  - Return f

<u>Theorem</u>. Ford-Fulkerson computes Max Flow correctly.

* FF Returns a valid flow.
* Flow f^* returned by FF s.t.

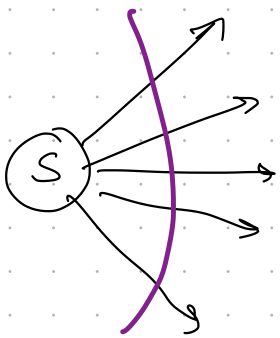
$$\text{FlowOut}^{f^*}(s) \geq \text{FlowOut}^{f}(s)$$

for all valid flows f.

New notation: for flow f, $\mathrm{val}(f) = \mathrm{FlowOut}^f(s)$.

$$= \sum_{sv \in E} f_{sv}$$

Fact. For all f, $\mathrm{val}(f) \leq \mathrm{Cap}(s) = \sum_{sv \in E} c_{sv}$.



$\sum_{sv \in E} c_{sv}$

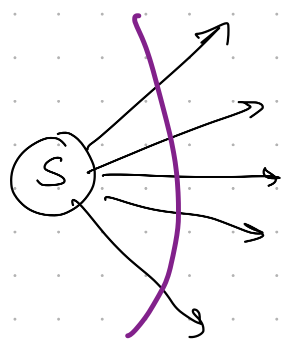
- All flow must leave s

- Can't push more than total capacity.

New notation: for flow f, $val(f) = FlowOut^f(s)$.

$$= \sum_{sv \in E} f_{sv}$$

<u>Fact</u>. For all f, $val(f) \leq Cap(s) = \sum_{sv \in E} c_{sv}$.



$$\sum_{sv \in E} c_{sv}$$

- All flow must leave (s)
- Can't push more than total capacity.

So $Cap(s)$ gives an upper bound on $val(f)$.

Can we find a tighter upper bound?

More generally.

Consider an st-Cut of G.

$$S \subseteq V \qquad T = V \setminus S$$

partition of vertices s.t.

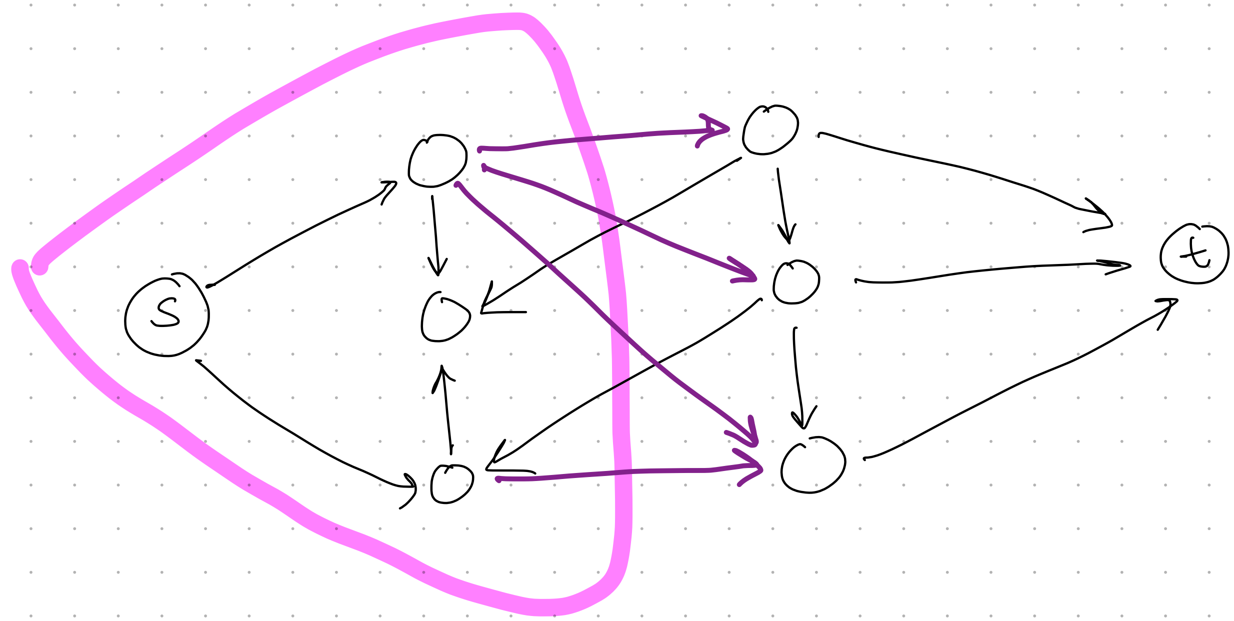
$$s \in S \qquad t \in T.$$

More generally.

Consider an st-Cut of G.

$S \subseteq V \qquad T = V \setminus S$

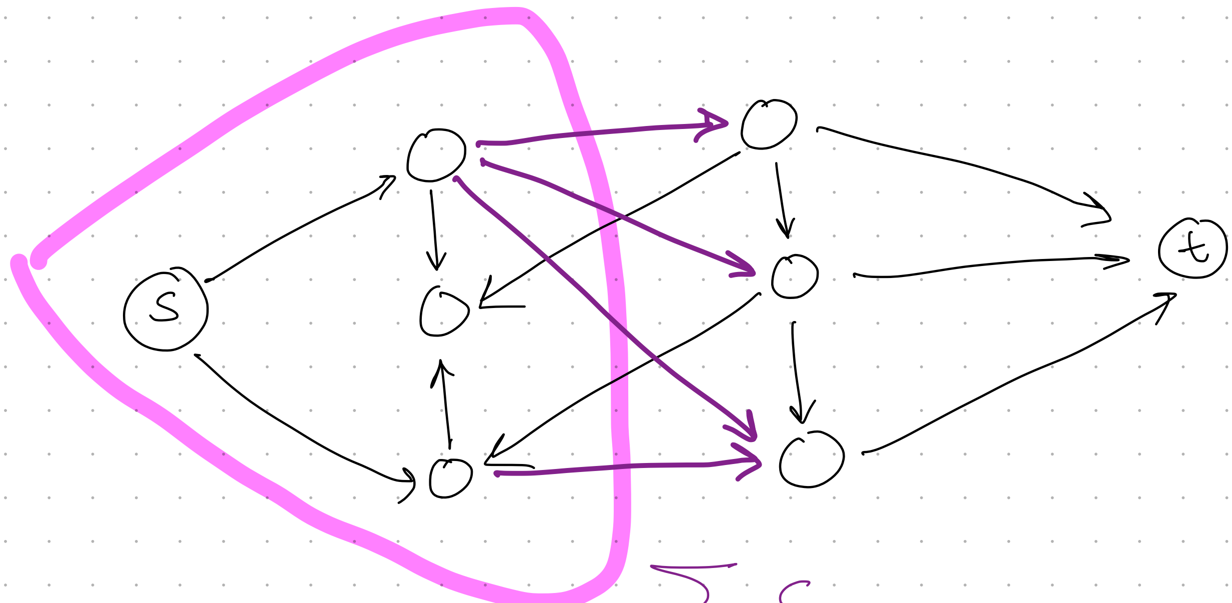
s.t. $s \in S \qquad t \in T$.



Capacity of an st-cut. $\mathrm{Cap}(S,T) = \underbrace{\sum_{\substack{uv \in E \\ u \in S,\, v \in T}}}_{\text{edges from } S \text{ to } T} c_{uv}$

Fact. For all f and any st-cut $S \subseteq V$

$$\mathrm{val}(f) \leq \mathrm{Cap}(S,T)$$



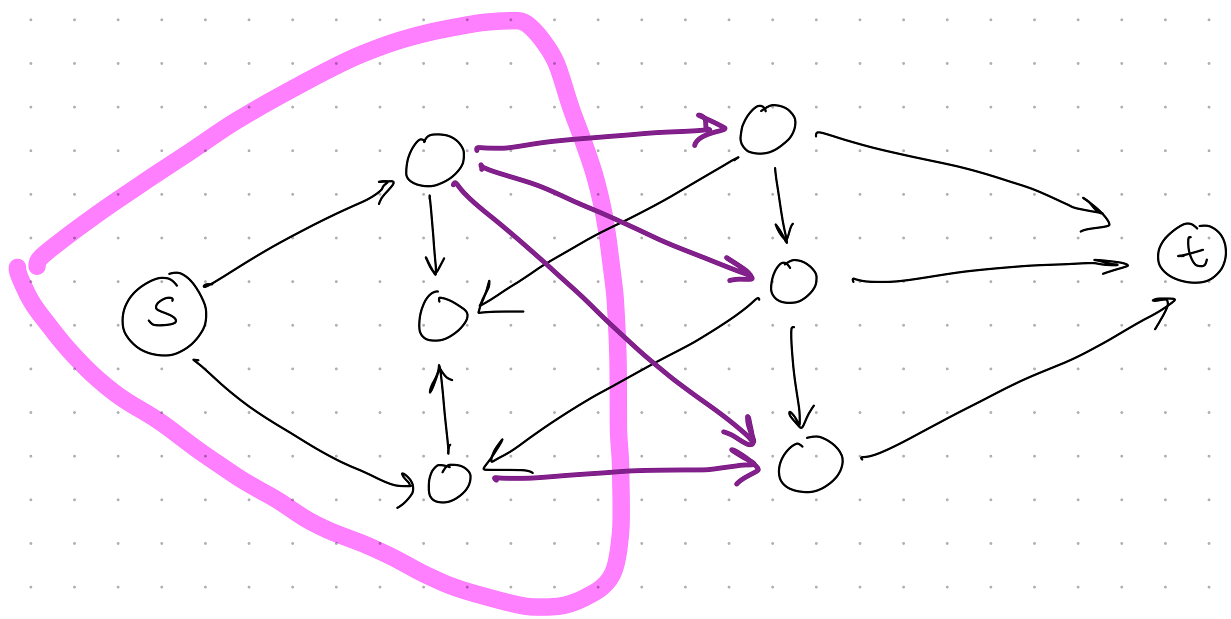
$$\sum_{\substack{uv \in E \\ u \in S,\, v \in T}} c_{uv}$$

- All flow must leave S
- Can't push more than total capacity leaving cut

So

For every flow network G, s, t, $c: E \to \mathbb{R}^+$

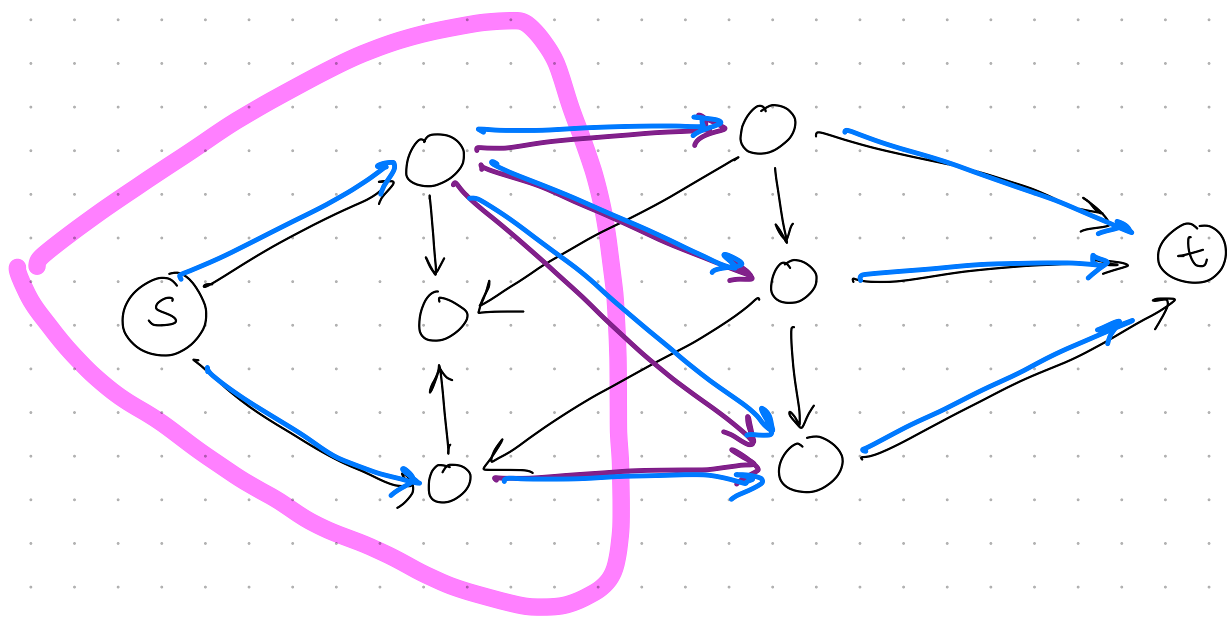
$$\max_{\text{flows } f} \mathrm{val}(f) \leq \min_{\substack{\text{st-cut} \\ S,T}} \mathrm{Cap}(S,T)$$



Theorem (Max Flow / Min Cut)

For every flow network G, s, t, $c : E \to \mathbb{R}^+$

$$\boxed{\max_{\text{flows } f} \text{val}(f) = \min_{\substack{st\text{-cut} \\ S,T}} \text{Cap}(S,T)}$$



Finding a maximum flow is equivalent to finding a minimum st-cut.

Proof by Analysis of Ford-Fulkerson.

Lemma. For any flow f and any st-Cut $S \subseteq V$.

$$val(f) = FlowOut^f(S) - FlowIn^f(S)$$

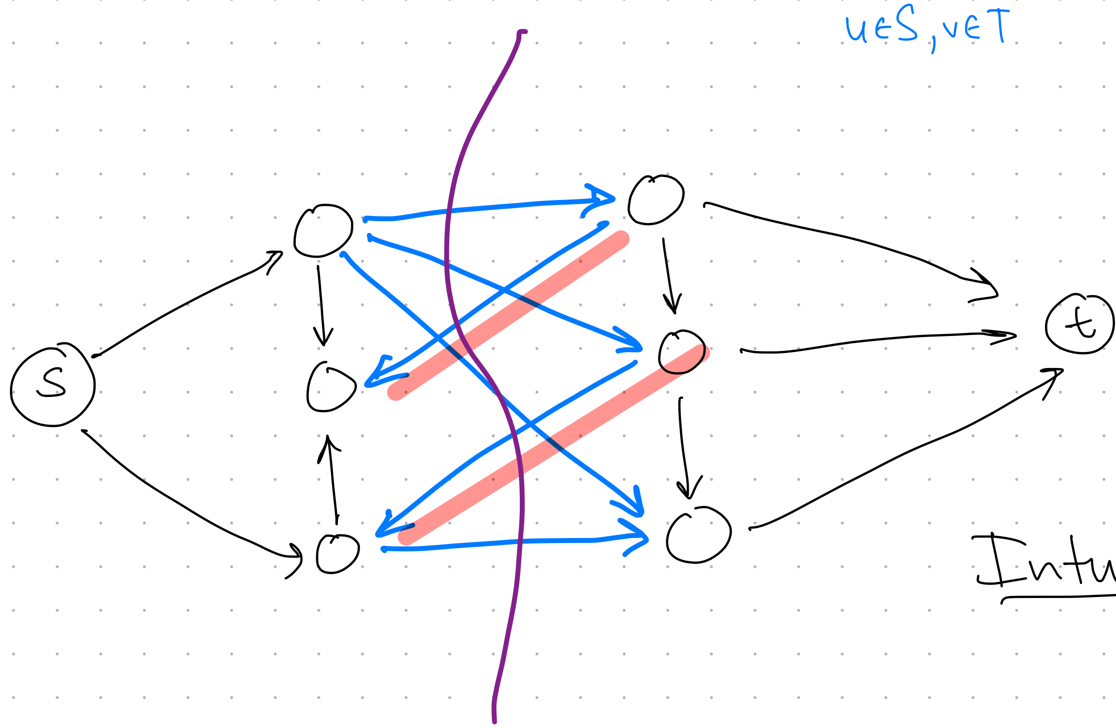
Claim. When FF terminates w/ f, there exists st-Cut S, T such that

$$val(f) = FlowOut^f(S) = Cap(S, T)$$

Lemma. For any st-Cut $S \subseteq V$ and any flow f.

$$\mathrm{val}(f) = \mathrm{FlowOut}^f(S) - \mathrm{FlowIn}^f(S)$$

$$= \sum_{u \in S, v \in T} f_{uv} - \sum_{v \in T, u \in S} f_{vu}$$



Intuition. Flows between $u, u' \in S$ "cancel"

Similar for $v, v' \in T$.

Net flow crosses cut.

Lemma. For any st-cut $S \subseteq V$, and any flow f.

$$\mathrm{val}(f) = \sum_{u \in S, v \in T} f_{uv} - \sum_{v \in T, u \in S} f_{vu}$$

Pf.

$$\begin{aligned}
\mathrm{val}(f) &= \mathrm{FlowOut}(s) \\
&= \mathrm{FlowOut}(s) + \sum_{u \in S \setminus \{s\}} \underbrace{\left(\mathrm{FlowOut}(u) - \mathrm{FlowIn}(u)\right)}_{0} \quad \leftarrow \text{CONSERVATION} \\
&= \sum_{u \in S} \left(\sum_{u' \in S} f_{uu'} + \sum_{v \in T} f_{uv} - \sum_{u' \in S} f_{u'u} - \sum_{v \in T} f_{vu} \right) \\
&= \underbrace{\sum_{u, u' \in S} \left(f_{uu'} - f_{uu'} \right)}_{0} + \sum_{u \in S, v \in T} f_{uv} - \sum_{u \in S, v \in T} f_{vu}
\end{aligned}$$

Claim. When FF terminates w/ f, there exists st-cut S, T such that

$$\text{val}(f) = \text{FlowOut}^f(S) = \text{Cap}(S, T)$$

Claim. When FF terminates w/ f, there exists st-Cut S, T such that

$$\text{val}(f) = \text{FlowOut}^f(S) = \text{Cap}(S, T)$$

Termination Condition: No st-path in G^f.

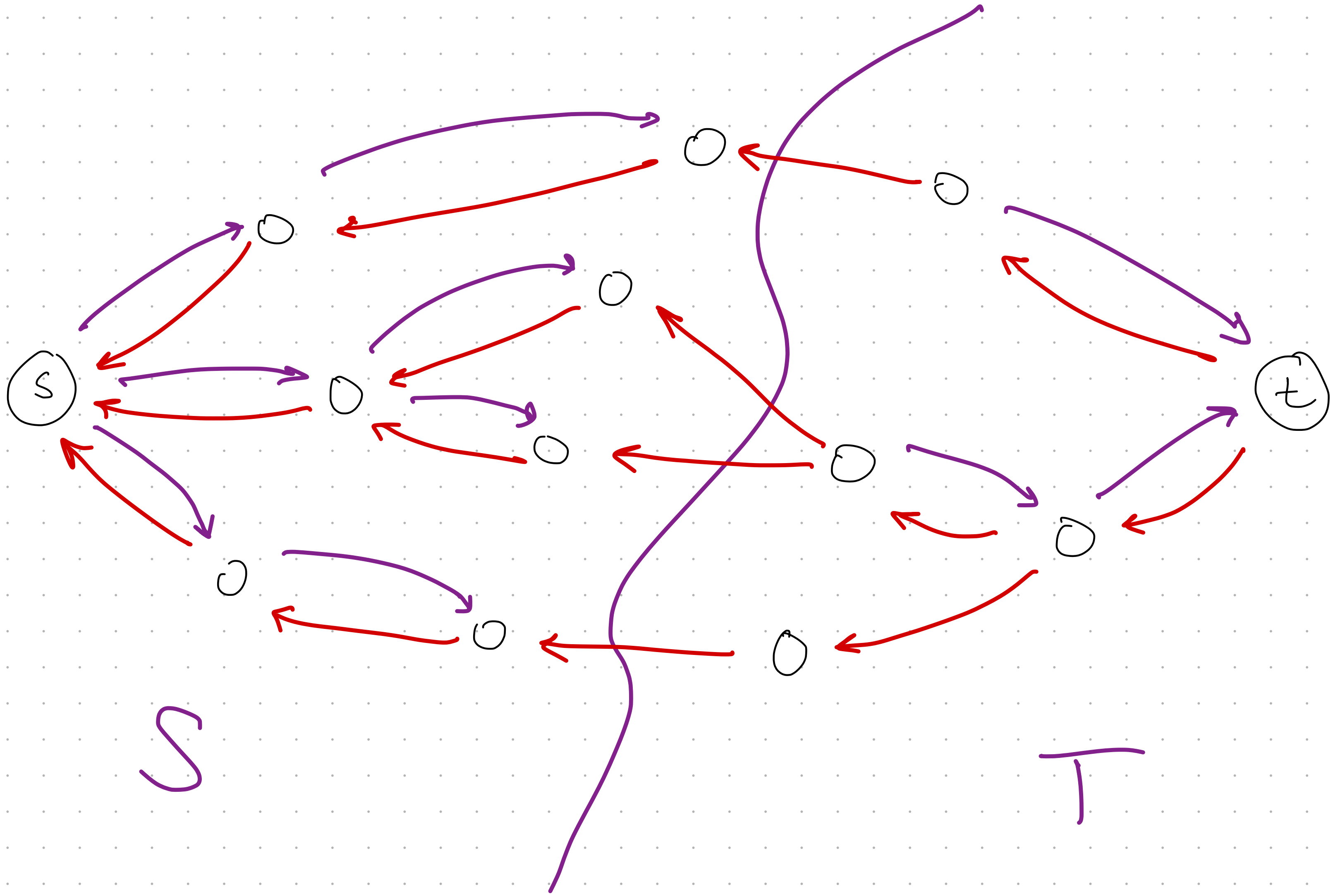
Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$

$T = V \setminus S$.

Note: $t \in T$ by termination condition.

Termination Condition: <u>No st-path in G^f.</u>

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$



Claim. When FF terminates w/ f, there exists st-cut S, T such that

$$val(f) = FlowOut^f(S) = Cap(S,T)$$

Termination Condition: No st-path in G^f.

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$

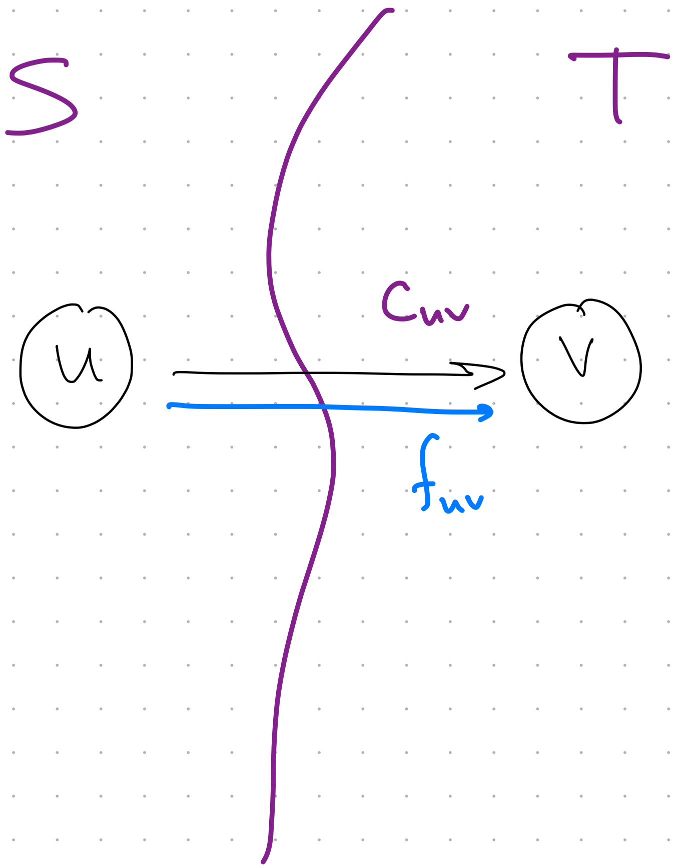
We show:

$\forall u \in S, v \in T$: $f_{uv} = c_{uv}$.

$\forall v \in T, u \in S$: $f_{vu} = \emptyset$.

establishes the Claim.

Define $S = \{ u \in V : u$ is reachable from s in $G^f \}$



Suppose $uv \in E$.

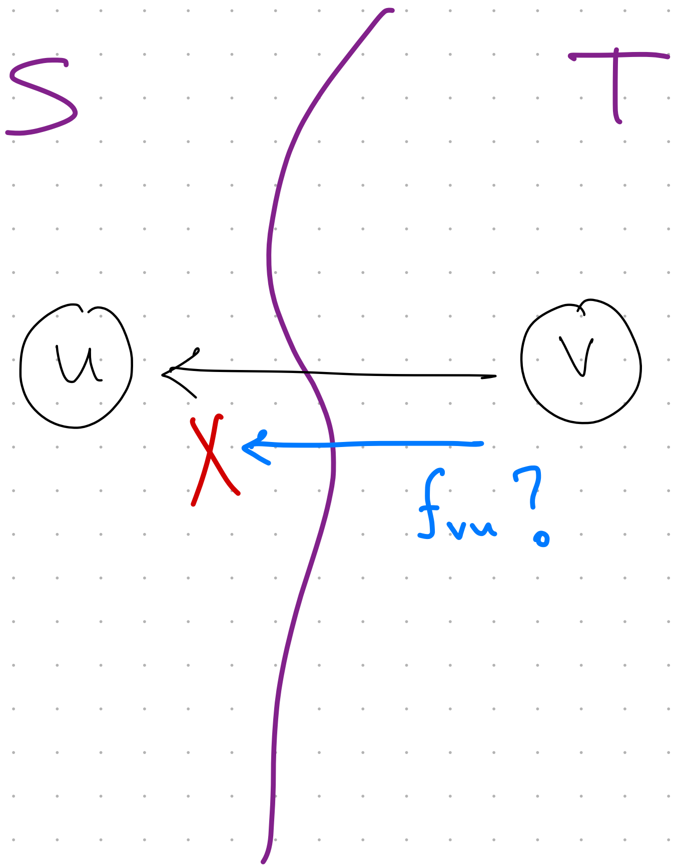
<u>Claim</u>. $f_{uv} = c_{uv}$.

<u>Pf</u>. Suppose $f_{uv} < c_{uv}$.

$\implies c^f(uv) = c_{uv} - f_{uv} > 0$

contradicting v not reachable from u.

Define $S = \{ u \in V : u \text{ is reachable from } s \text{ in } G^f \}$



Suppose $vu \in E$.

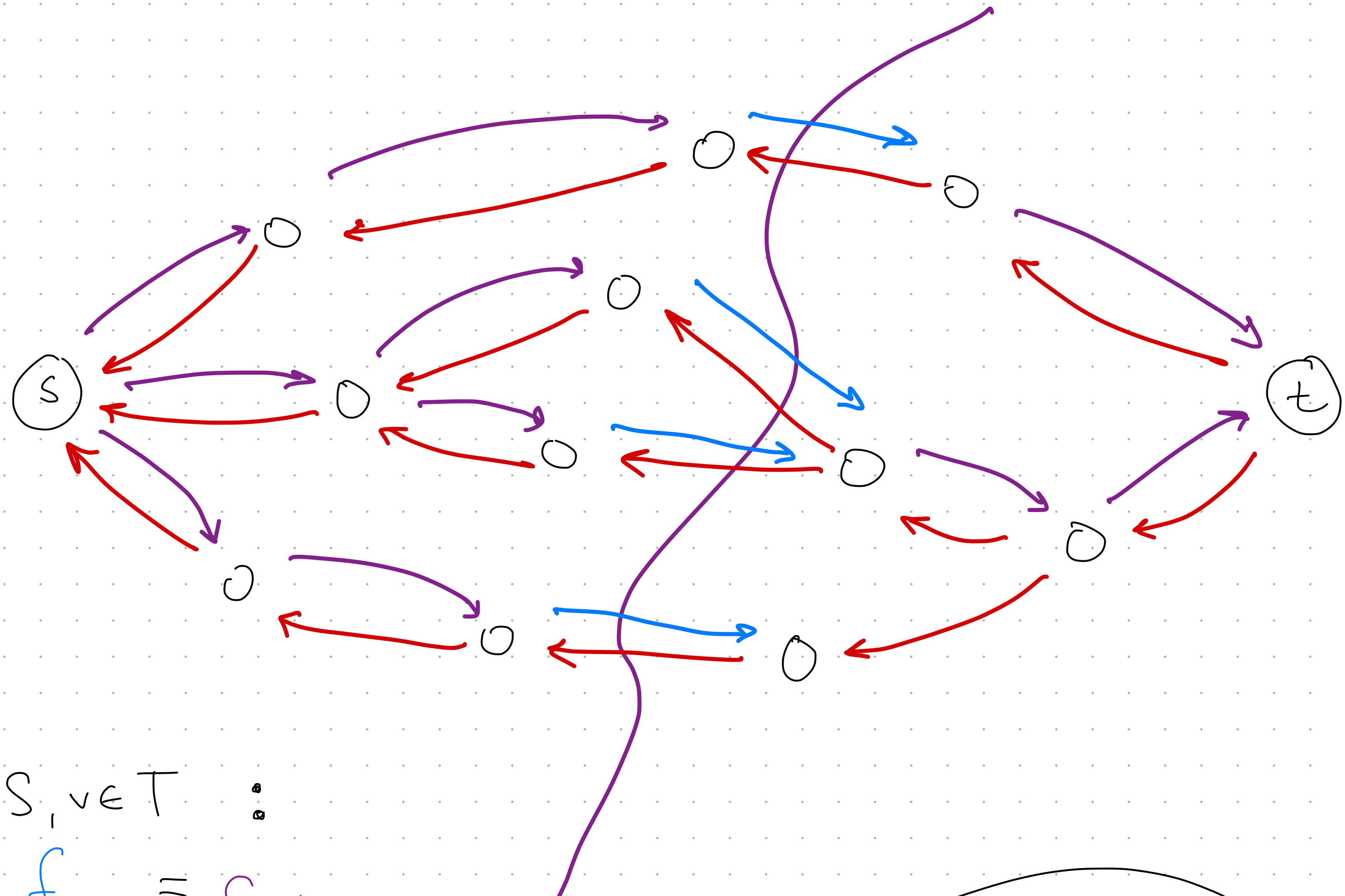
<u>Claim</u>. $f_{uv} = 0$.

<u>Pf</u>. Suppose $f_{uv} > 0$

$\implies c^f(uv) = f_{uv} > 0$

contradicting v not reachable from u.

Termination Condition: No st-path in G^f.



$\forall u \in S, v \in T$:

$$f_{uv} = c_{uv}.$$

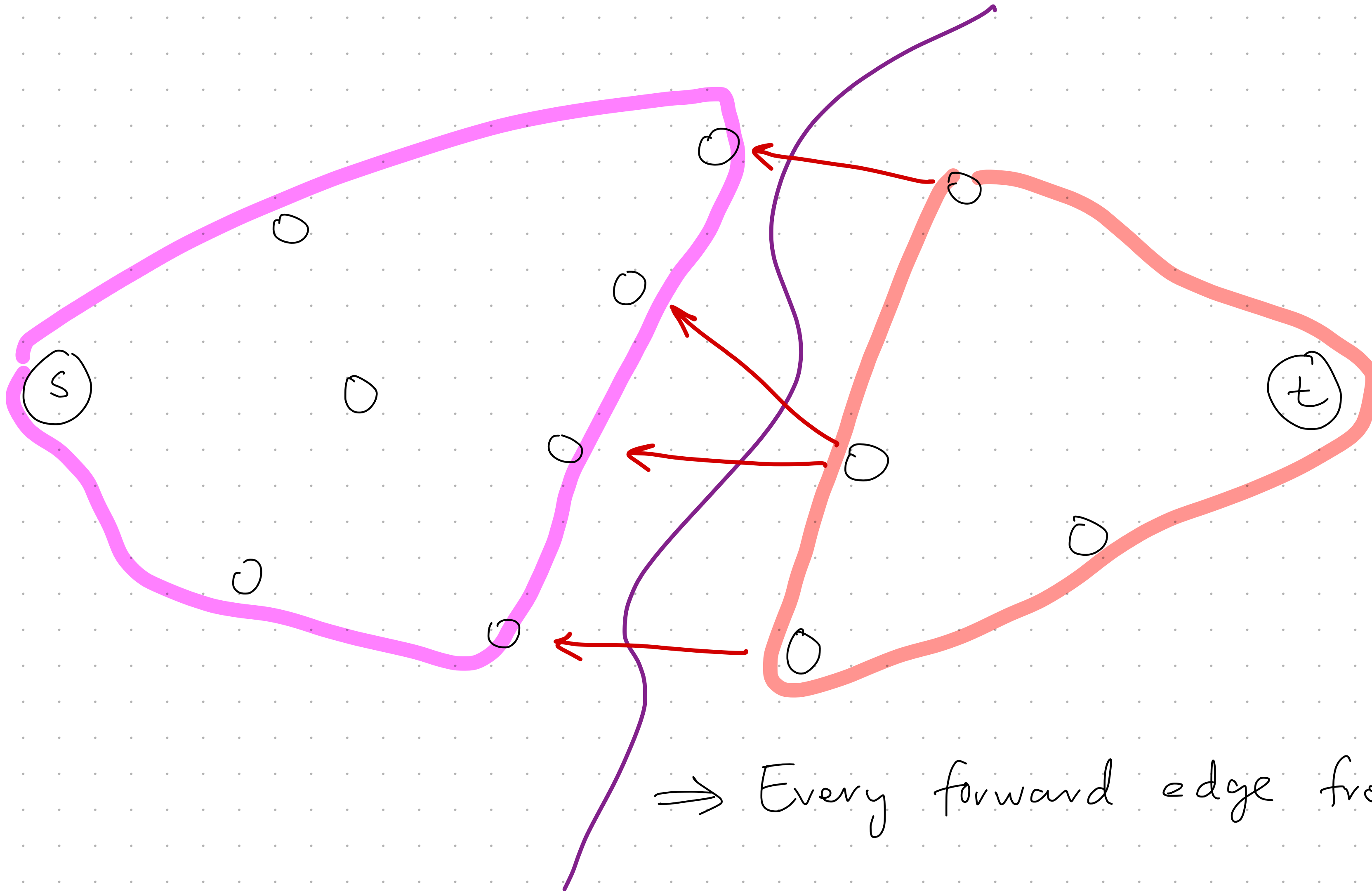
$\forall v \in T, u \in S$:

$$f_{vu} = \emptyset.$$

$$val(f) = \sum_{u \in S, v \in T} f_{uv} - \underbrace{\sum_{v \in T, u \in S} f_{vs}}_{\emptyset} = \sum_{u \in S, v \in T} c_{uv}$$

Termination Condition: No st-path in G^f.

↳ only backward residual edges from T to S



⟹ Every forward edge from S → T saturated

& No flow from T → S

By Max Flow / Min Cut Theorem

S, T defined by Ford-Fulkerson Termination is a min cut.