21 February 2025

Maximum Bipartite Matching

Plan

* Max Bipartite Matching
* Announcements
* Reduction to Flow.

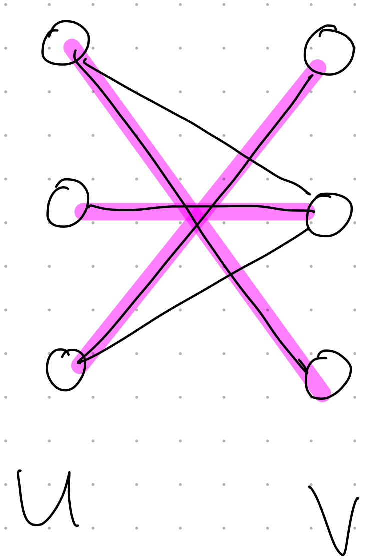
Maximum Bipartite Matching

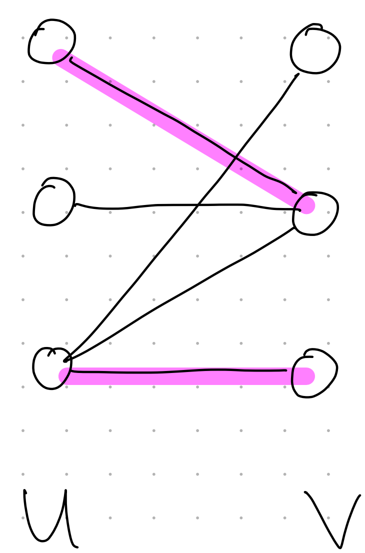
Given. Bipartite Graph on vertices $U \cup V$.

Every edge in E has exactly 1 endpoint in U and V.

Find. Maximum Cardinality Matching $M \subseteq E$.

Collection of edges s.t. every vertex is in at most one edge.





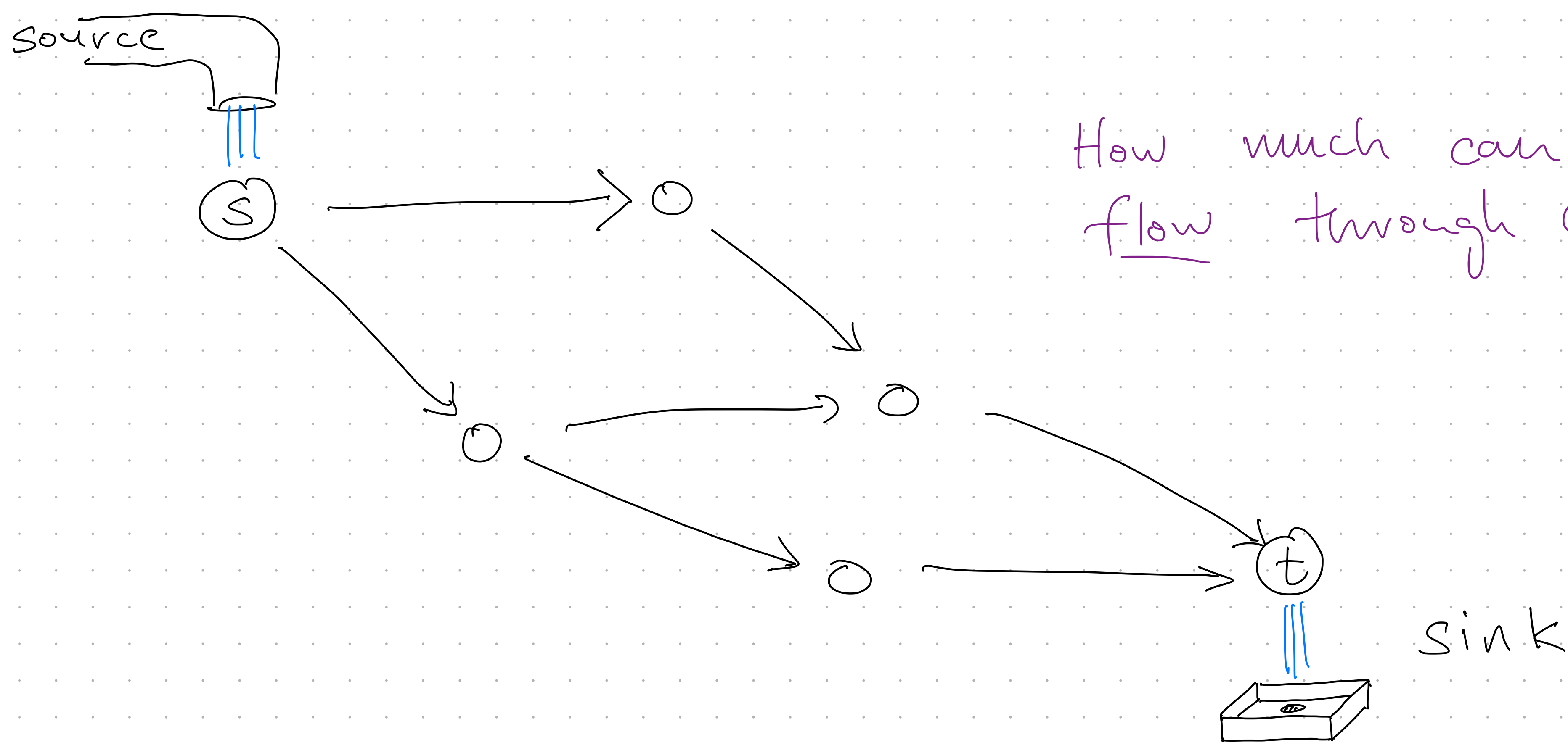
Network Flow Problem

Given: Directed graph $G = (V, E)$

Source vertex s

Sink vertex t

Edge capacities $c_e \geq 0 \quad \forall e \in E$.



How much can flow through G?

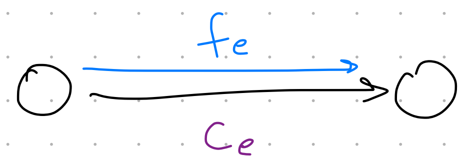
Network Flow Problem

* Given Flow Network $G, s, t, c: E \to \mathbb{R}^+$
* Find Flow $f: E \to \mathbb{R}^+$ subject to

Capacity Constraints

$\forall e \in E$

$$0 \le f_e \le c_e$$



Conservation Constraints

$\forall v \in V \setminus \{s,t\}$

$$\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw}$$

Max Flow. Find Flow f^* that maximizes FlowOut(s)

Announcements

* Grading Ongoing
  - ↳ Prelim 1
  - ↳ HW2

* No Recitation this Saturday.

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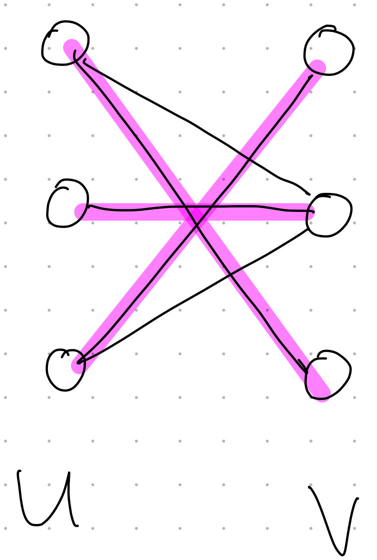
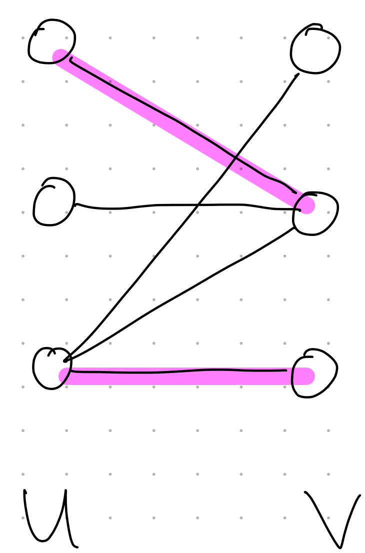


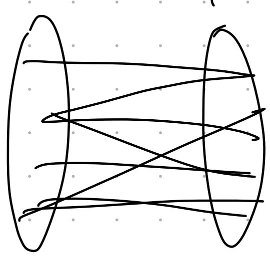


Reducing Max Bipartite Matching to Max Flow

* Reduction is an algorithm R.

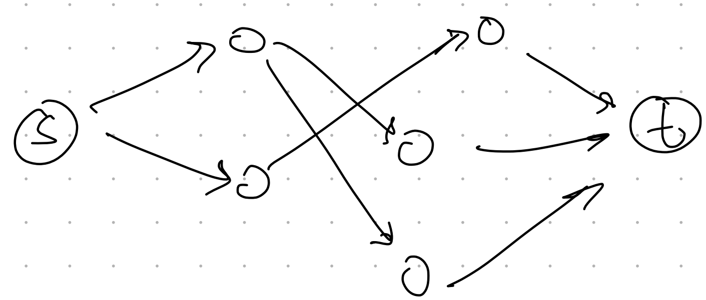
$$\text{Bipartite Matching instances} \xrightarrow{R} \text{Max Flow instances}$$

Bipartite Graph
$B = (U, V, E)$



B

Flow Network
$R(B) = (G, s, t, c)$



R(B)

Reducing Max Bipartite Matching to Max Flow

* Reduction is an algorithm R.

$$\text{Bipartite Matching instances} \xrightarrow{R} \text{Max Flow instances}$$

B

R(B)

Conclusion. If we can solve Max Flow, then we can solve Max Bipartite Matching.

$$\text{MaxBipartiteMatching} \leq \text{MaxFlow}$$

Reducing Max Bipartite Matching to Max Flow

* Reduction is an algorithm R.

$$\text{Bipartite Matching instances} \xrightarrow{R} \text{Max Flow instances}$$

$$B \qquad\qquad R(B)$$

* Requirements.

$$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$$

Reducing Max Bipartite Matching to Max Flow

* Reduction is an <u>algorithm</u> R.

$$\text{Bipartite Matching instances} \xrightarrow{R} \text{Max Flow instances}$$

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* <u>Requirements.</u>

($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$
then max flow in $R(B)$ is at least k.

Reducing Max Bipartite Matching to Max Flow

* Reduction is an algorithm R.

$$\text{Bipartite Matching instances } B \xrightarrow{R} \text{Max Flow instances } R(B)$$

* Requirements.

(⇒) If B has a matching M s.t. $|M| \geq k$
then max flow in $R(B)$ is at least k.

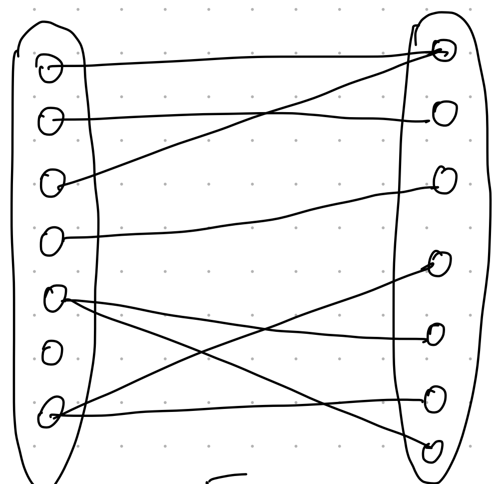
(⇐) If $R(B)$ has max flow at least k,
then B has a matching M s.t. $|M| \geq k$.

Reduction

On input $B = (U, V, E)$

Construct G on vertices $U \cup V \cup \{s, t\}$

B



E

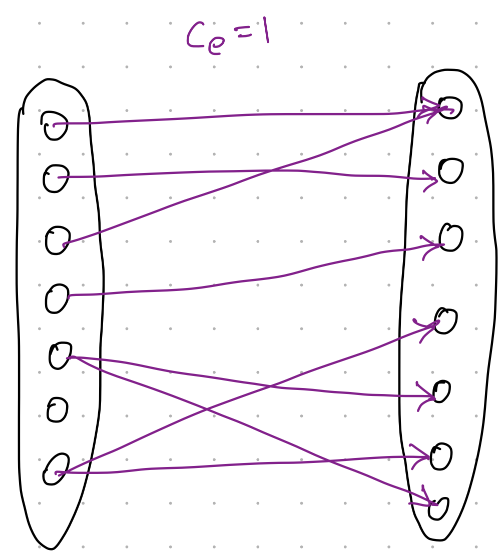
U V

Reduction

On input $B = (U, V, E)$

Construct G on vertices $U \cup V \cup \{s, t\}$

$\hookrightarrow$ For every $(u,v) \in E$, add directed edge to G w/ capacity $c_{uv} = 1$.



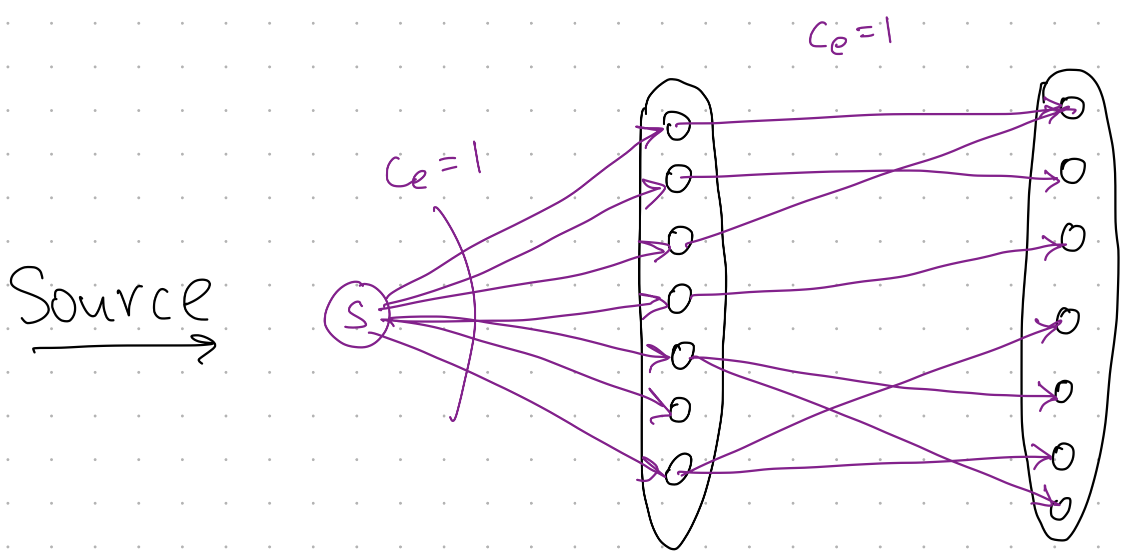
Reduction

On input $B = (U, V, E)$

Construct G on vertices $U \cup V \cup \{s, t\}$

↳ For every $(u,v) \in E$, add directed edge to G w/ capacity $c_{uv} = 1$.

↳ For every $u \in U$, add (s,u) to G w/ cap $c_{su} = 1$



Reduction.

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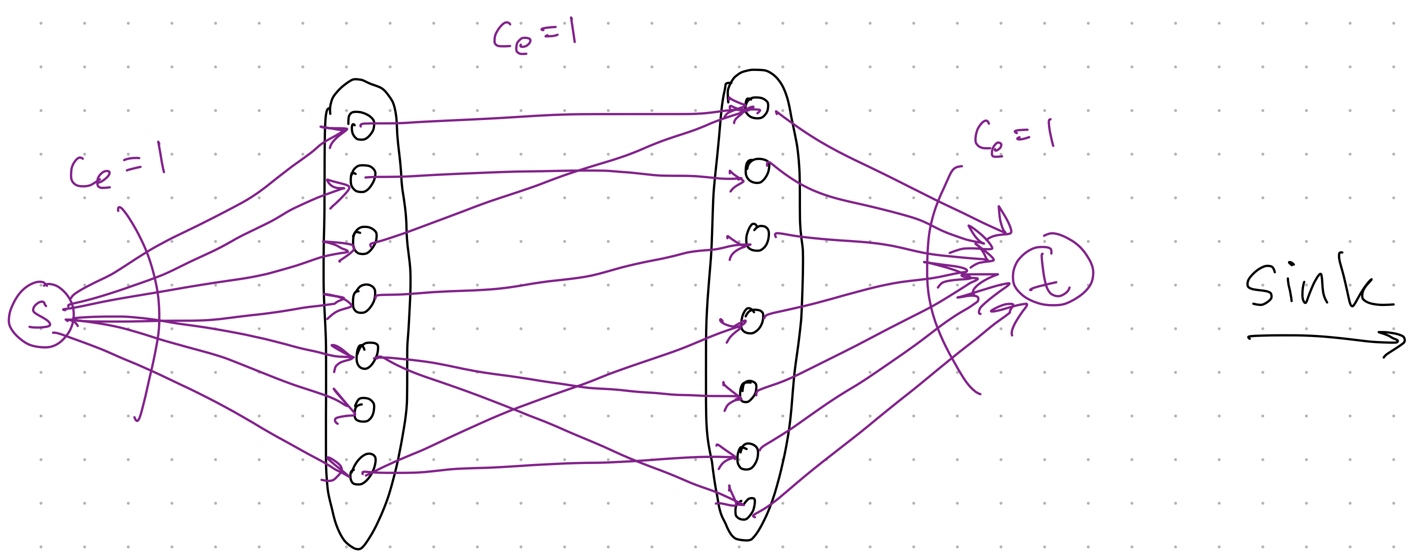
$R(B)$



Max Bipartite Match Algo.

On input $B = (U, V, E)$

Run Reduction to Max Flow $R(B)$

Compute $k \leftarrow \text{MaxFlow}(R(B))$

Return k. // Returns cardinality of Max Matching in B.

Max Bipartite Match Algo.

On input $B = (U, V, E)$

Run Reduction to Max Flow $R(B)$

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Running Time.

Time of Reduction $\rightarrow O(|U|+|V|+|E|)$

Time to Solve Max Flow

$\rightarrow MF(O(n), O(m+n))$

nodes edges

Proof of Correctness.

* Need to show

$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$

* What can we leverage?

Proof of Correctness.

* Need to show

$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$

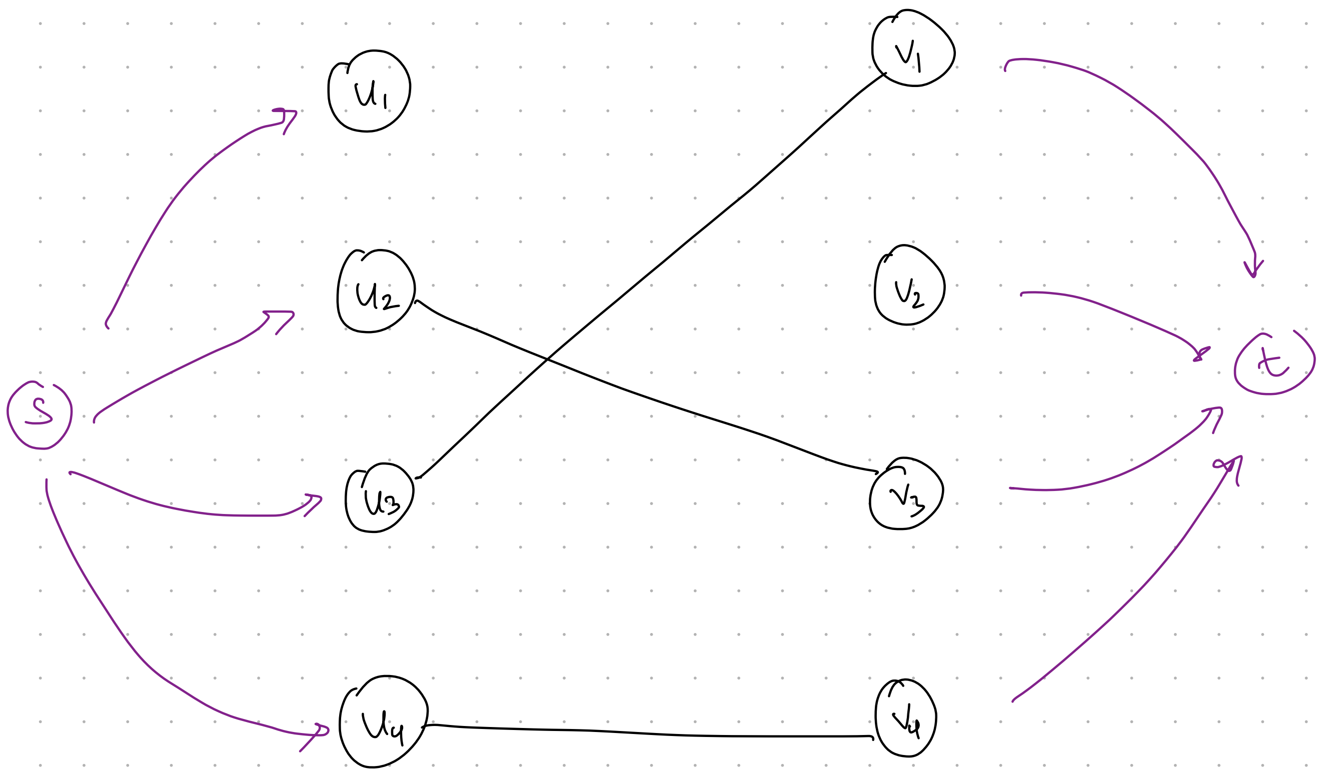
* What can we leverage?
  - Defn of matching → $\forall u \in U$, $v \in V$ in at most one pair in M
  - Flow constraints
    - CAPACITY → $0 \leq f_e \leq c_e \quad \forall e \in E$
    - CONSERVATION → $\text{FlowIn}(v) = \text{FlowOut}(v) \quad \forall v \in V$
  - Properties of Reduction R

FORWARD DIRECTION

To show.

($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$,

then max flow in $R(B) \geq k$.



($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$,
then max flow in $R(B) \geq k$.

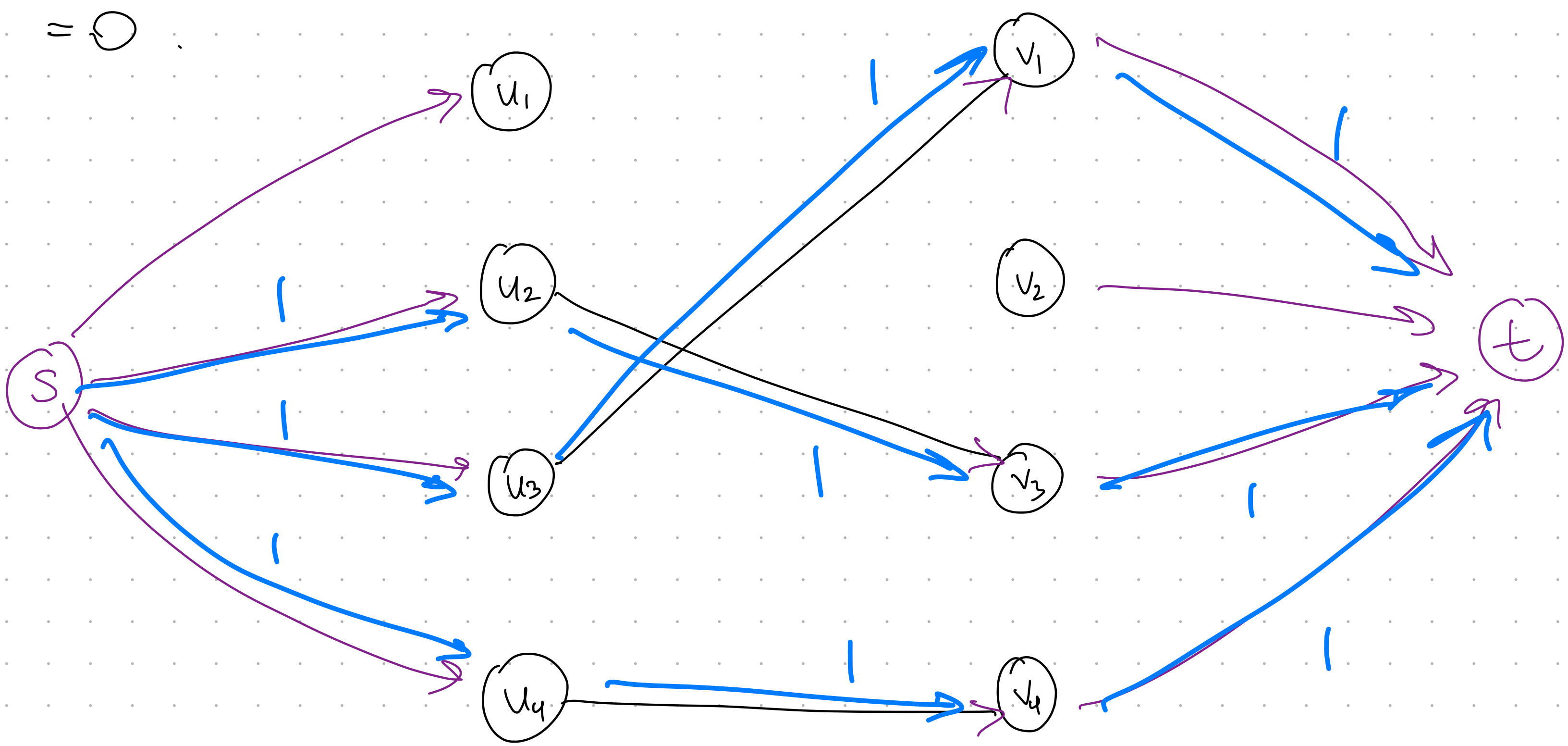
Consider routing 1 unit of flow along each matched edge $(u,v) \in M$

For all $(u,v) \in M$

$$f_{su} = f_{uv} = f_{vt} = 1$$

For all other edges

$$f_e = 0.$$



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Claim 1. $\text{FlowOut}(s) = |M|$

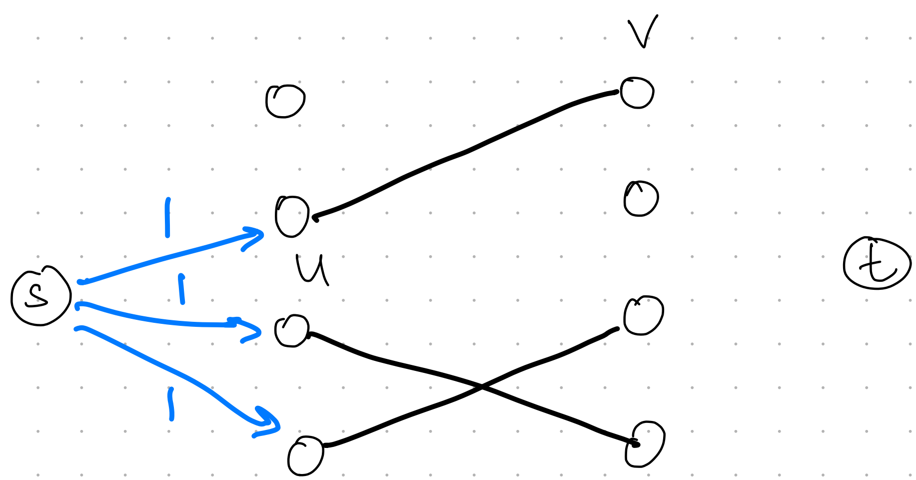
Claim 2. f is a feasible flow.

Claim 1. FlowOut(s) = |M|

By defn. of matching,

every $u \in U$ involved in at most 1 matched edge

$\Rightarrow$ |M| distinct endpoints $u \in U$ w/in M.



$\Rightarrow$ FlowOut(s) = |M|

Claim 2. f is a feasible flow

* CAPACITY CONSTRAINTS
* CONSERVATION CONSTRAINTS

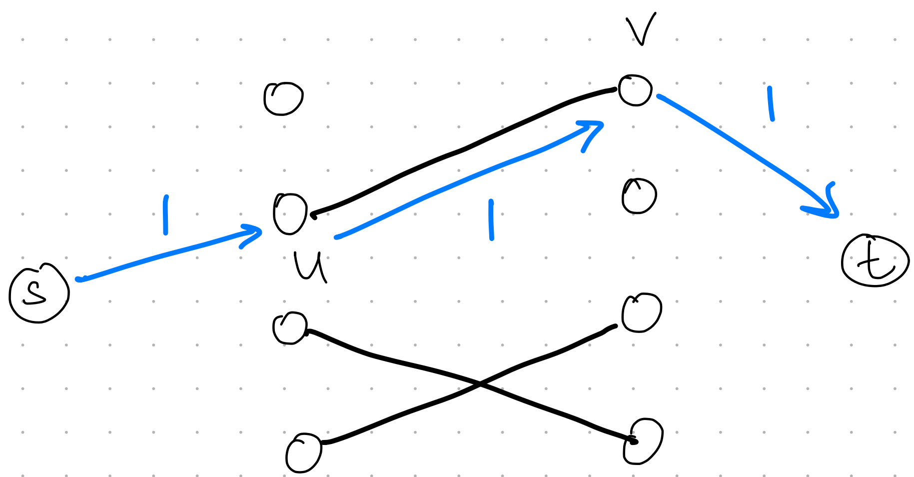
Claim 2. f is a feasible flow

* CAPACITY CONSTRAINTS

By defn. of matching,

$u \in U$ and $v \in V$ involved in at most 1 matched edge

$\Rightarrow \forall u \in U \quad f_{su} \leq 1 = c_{su} \quad \& \quad \forall v \in V \quad f_{vt} \leq 1 = c_{vt}$

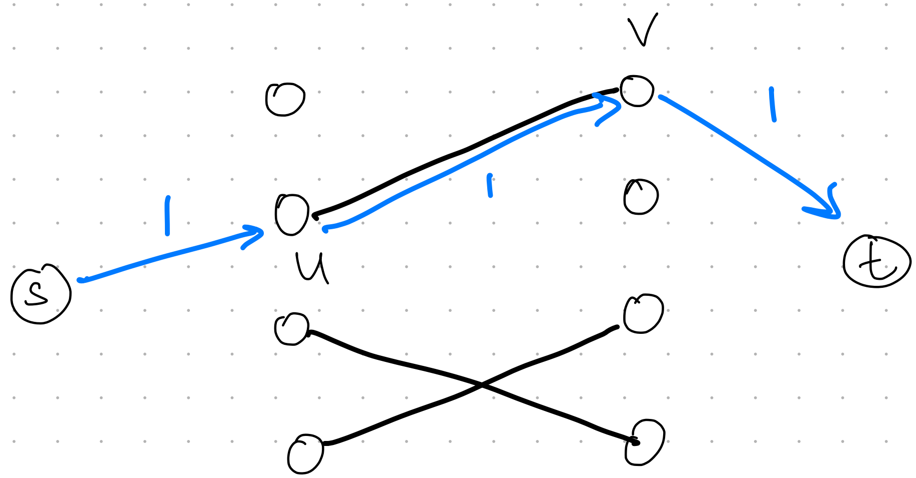


$\Rightarrow$ Capacity constraints on $s \to u$ and $v \to t$ OK ✓

* Only 1 unit routed on $f_{uv} \leq 1$ $\quad u \to v$ OK ✓

Claim 2. f is a feasible flow

* CAPACITY CONSTRAINTS
* CONSERVATION CONSTRAINTS ✓



* For every matched $(u,v) \in M$

$$s \xrightarrow{1} u \xrightarrow{1} v \xrightarrow{1} t$$

exactly 1 unit in/out of u & v.

Consider routing 1 unit of flow along each matched edge $(u,v) \in M$

Claim 1. $\text{FlowOut}(s) = |M|$ ✓

Claim 2. f is a feasible flow. ✓

($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$, then max flow in $R(B) \geq k$.

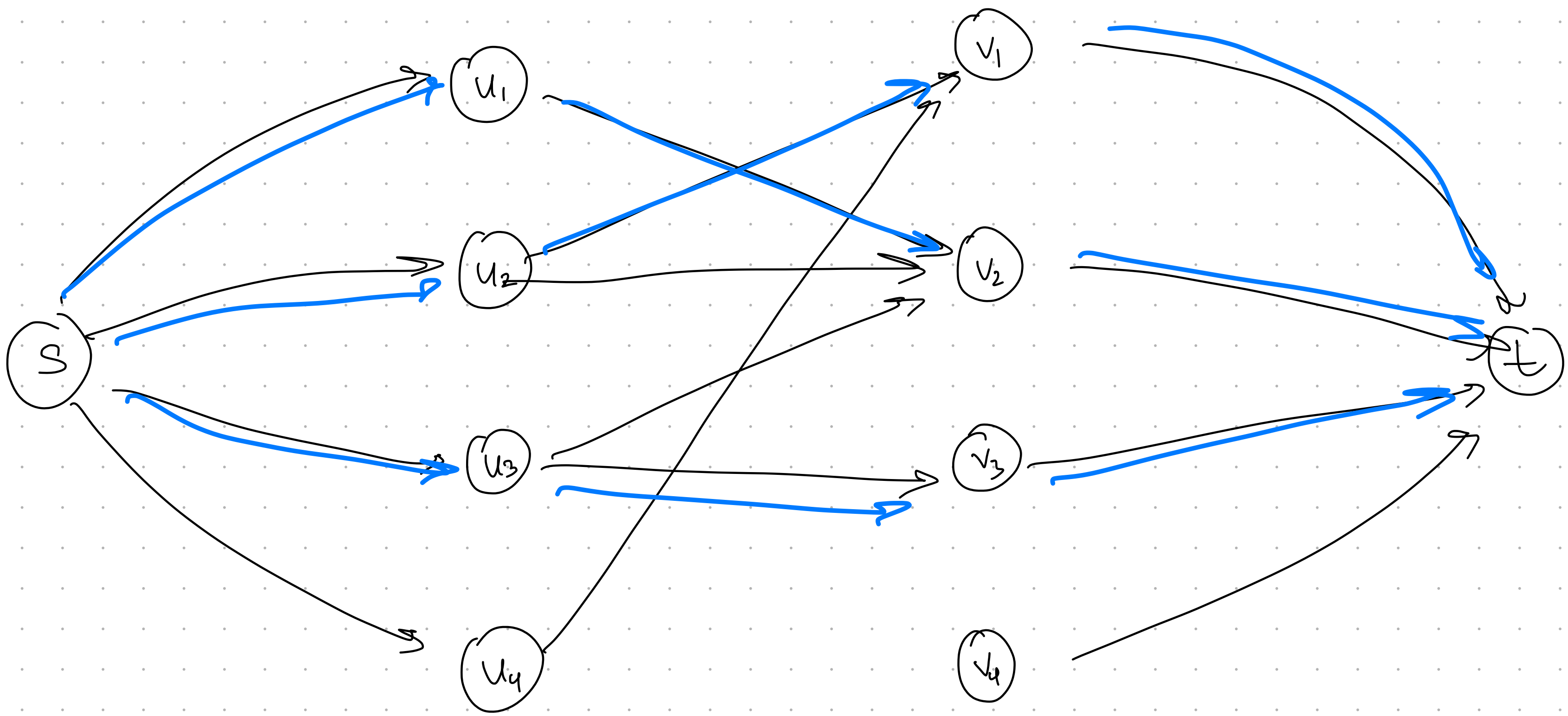
∎

BACKWARD DIRECTION

To show.

($\Leftarrow$) If max flow in $R(B) \geq k$,

then max matching in $B \geq k$.



($\Leftarrow$) If max flow in $R(B) \geq k$, then max matching in $B \geq k$.

<u>Fact</u>. If all capacities are integer, $\exists$ a max flow s.t. $f_e \in \mathbb{N}$ $\forall e \in G$

Consider such an integral max flow f.

$\Longrightarrow$ By capacities $\Longrightarrow$ $f_e \in \{0,1\}$ $\forall e \in G$

($\Leftarrow$) If max flow in $R(B) \geq k$,
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Consider such an integral max flow f.
$\Rightarrow$ By capacities $\Rightarrow$ $f_e \in \{0,1\}$ $\forall e \in G$

$$\boxed{\text{Let } M = \{(u,v) : u \in U,\ v \in V \text{ and } f_{uv} = 1\}}$$

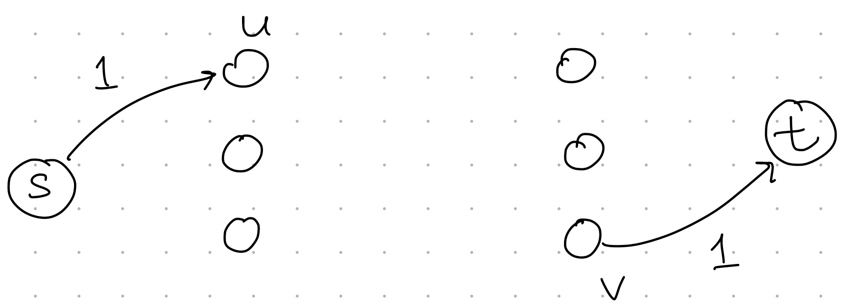
We prove M is a matching $|M| = \text{FlowOut}(s)$

Let $M = \{ (u,v) : u \in U, v \in V \text{ and } f_{uv} = 1 \}$

Claim. M is a matching.

Pf. By construction in Reduction R

* every $u \in U$ has exactly one in-edge of cap $c_{su} = 1$
* every $v \in V$ has exactly one out-edge of cap $c_{vt} = 1$



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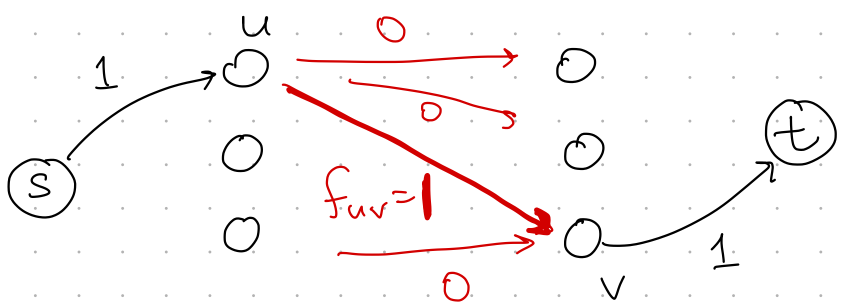

By conservation constraints.

$\Rightarrow$ At most 1 unit of flow out of every $u \in U$

$\Rightarrow$ At most 1 unit of flow into every $v \in V$.

Let $M = \{(u,v) : u \in U, v \in V$ and $f_{uv} = 1\}$

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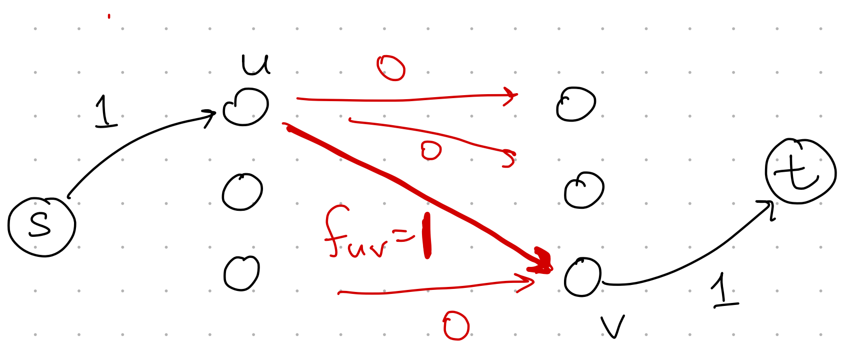


By conservation constraints,

$\Rightarrow$ At most 1 unit of flow out of every $u \in U$

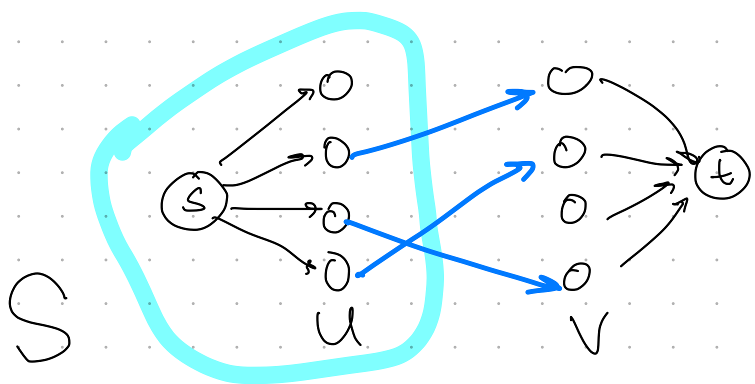
$\Rightarrow$ At most 1 unit of flow into every $v \in V$.

$\Rightarrow$ Every $u \in U$ and $v \in V$ involved in at most one edge in M ✓

Let $M = \{ (u,v) : u \in U, v \in V \text{ and } f_{uv} = 1 \}$

<u>Claim.</u> $|M| = \text{FlowOut}(s)$

<u>Pf.</u> Consider the cut in G, $S = \{s\} \cup U$.



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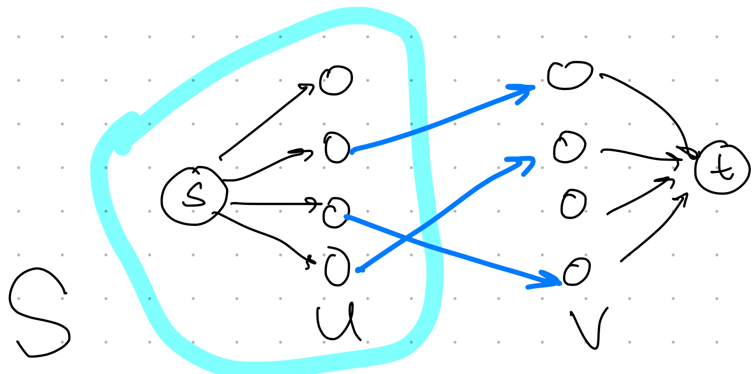

Note. No edges are oriented into S

$\Longrightarrow \text{FlowOut}(s) = \text{Flow out of } S$

$= \text{Flow from } u \in U \text{ to } v \in V.$

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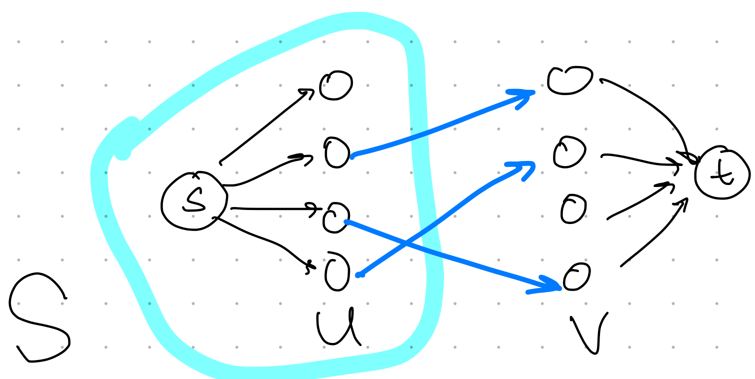

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$= \text{Flow from } u \in U \text{ to } v \in V.$

By integrality of $f_e \in \{0,1\} \Longrightarrow \geq k$ edges have non-zero flow from U to V.

$\Longrightarrow |M| = \text{FlowOut}(s)$ ✓

Next Question.

Can we actually solve Max Flow?