19 February 2025

Flow Networks

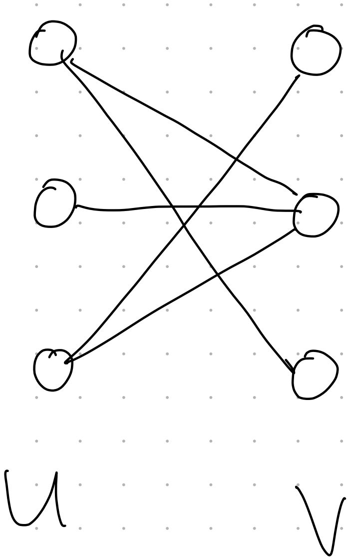
Plan

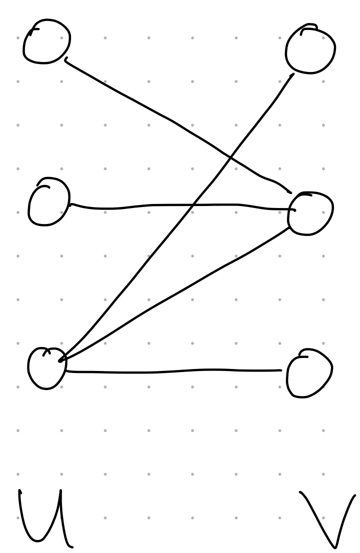
* Flow Problem
* Announcements
* Max Bipartite Matching: Reduction to Flow.

Maximum Bipartite Matching

Given. Bipartite Graph on vertices $U \cup V$.

Every edge in E has exactly 1 endpoint in U and V.





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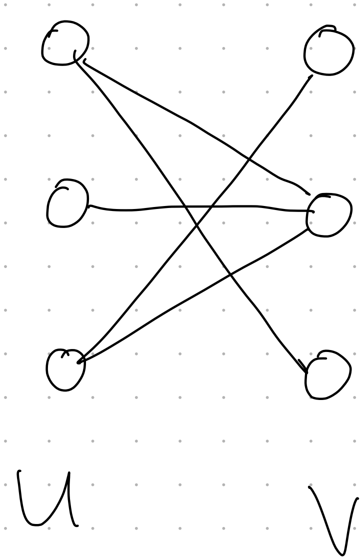
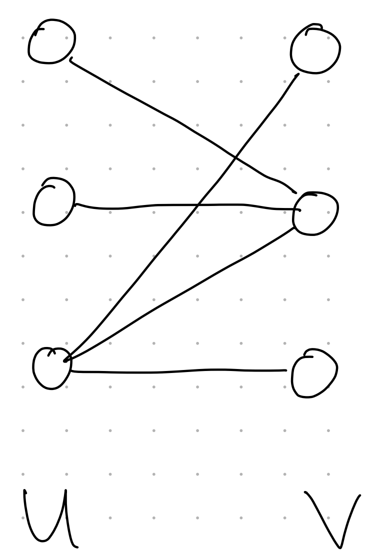
Find. Maximum Cardinality Matching $M \subseteq E$.

Collection of edges s.t. every vertex is in at most one edge.





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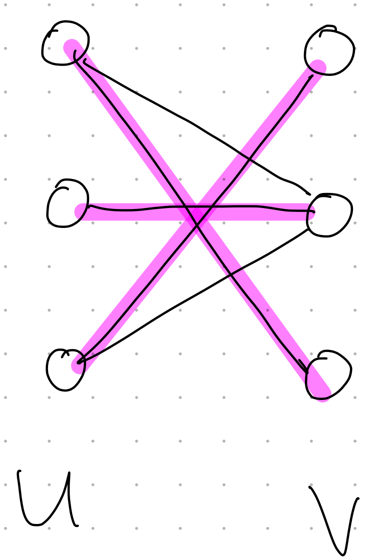

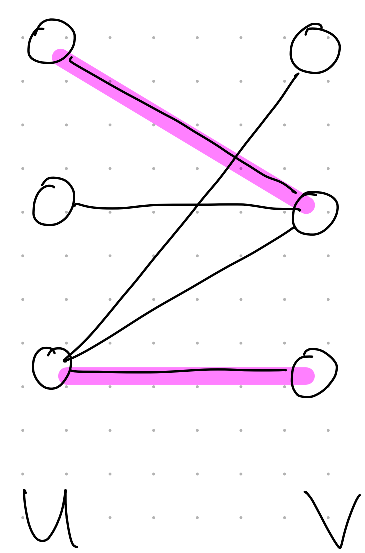

Questions

* How do we find a feasible matching $M \subseteq E$?

* How can we certify M is best possible?

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* How can we certify M is <u>best</u> possible?

Answers

Reduction to Network Flow

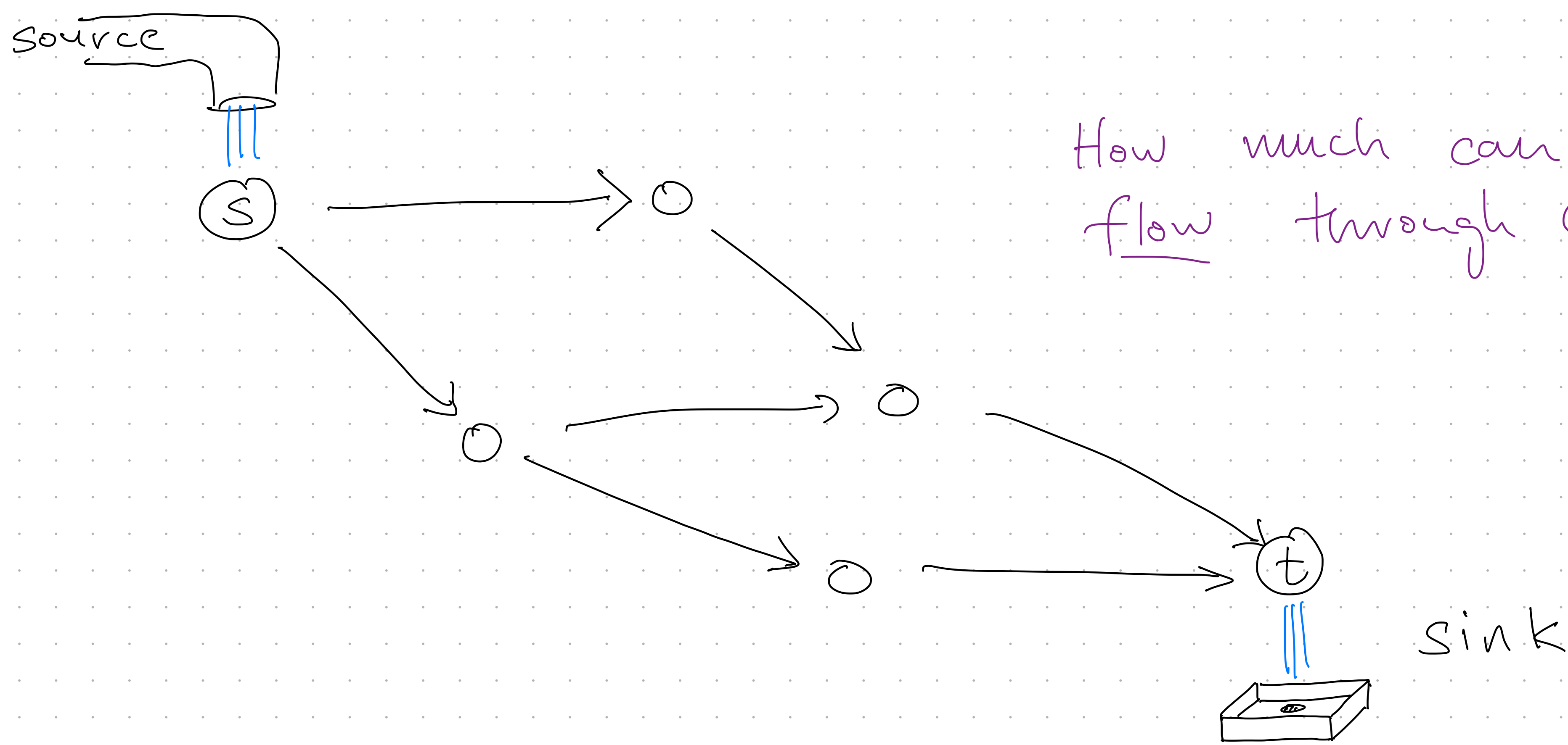
Network Flow Problem.

Given. Directed graph $G = (V, E)$

Source vertex s

Sink vertex t

Edge capacities $c_e \geq 0 \ \forall e \in E$.



How much can flow through G?

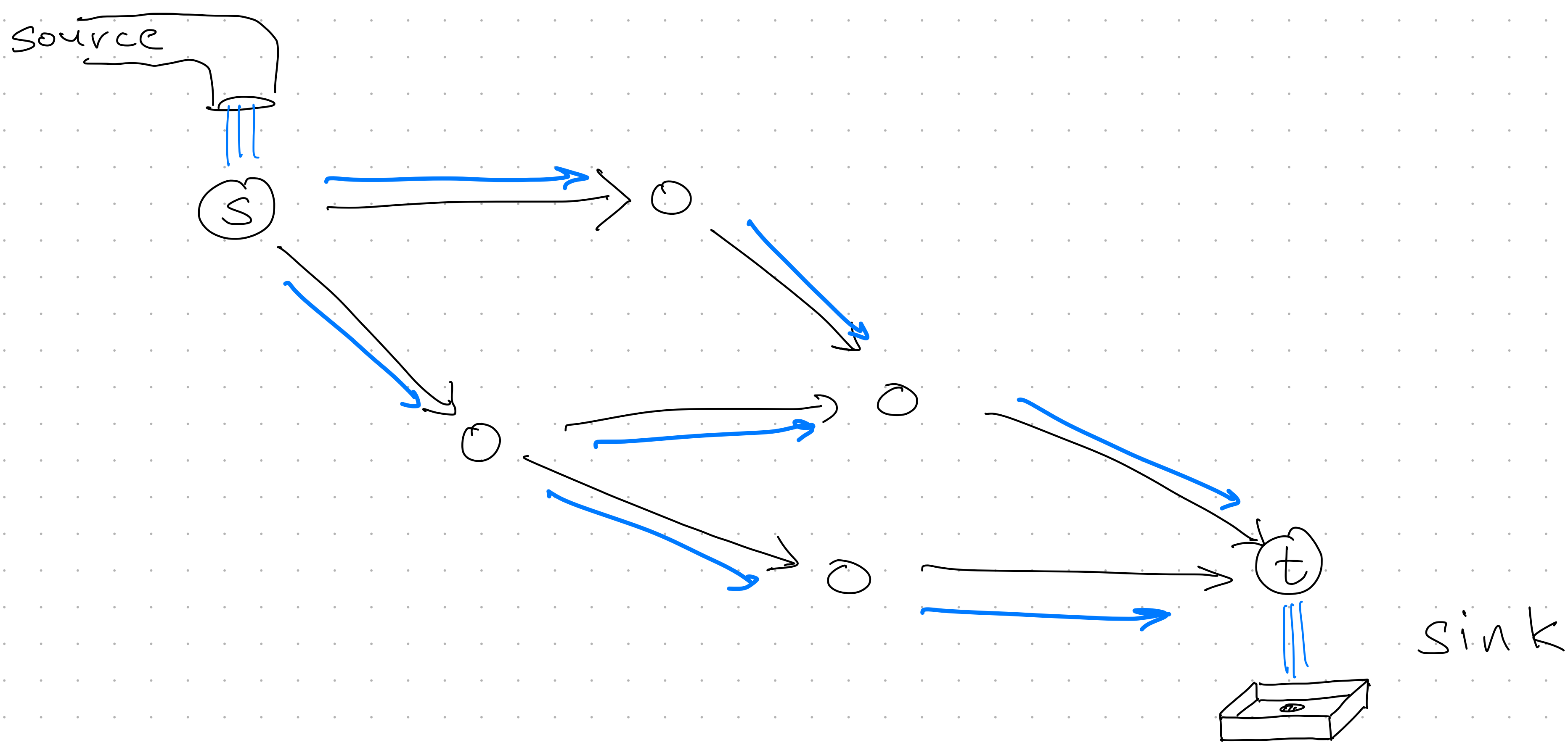
Network Flow Problem

Flow f

* Assigns quantity

$$f_e \geq 0$$

to each edge.



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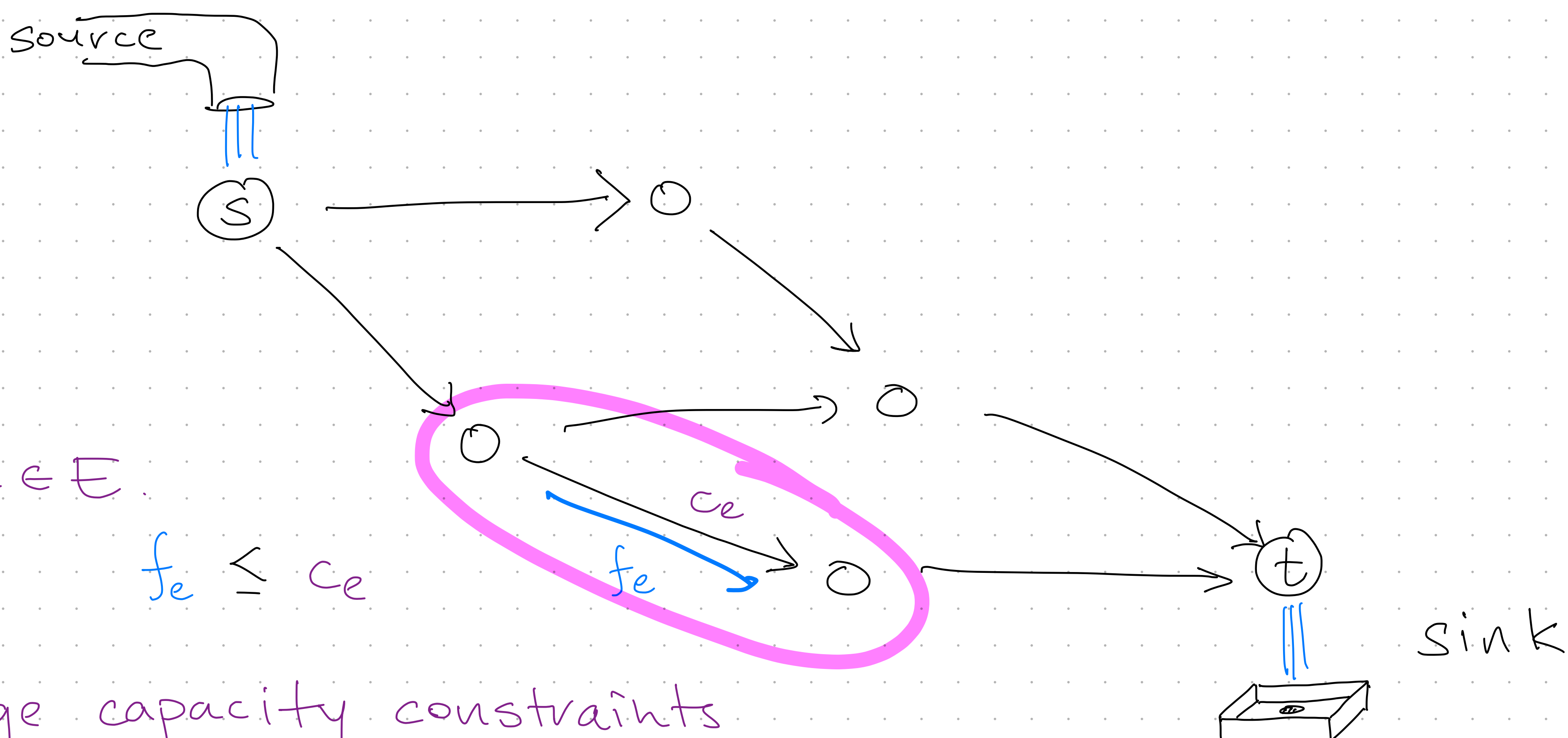

$\forall e \in E$.

$$f_e \leq c_e$$

Edge capacity constraints

Network Flow Problem

Flow f

* Assigns quantity
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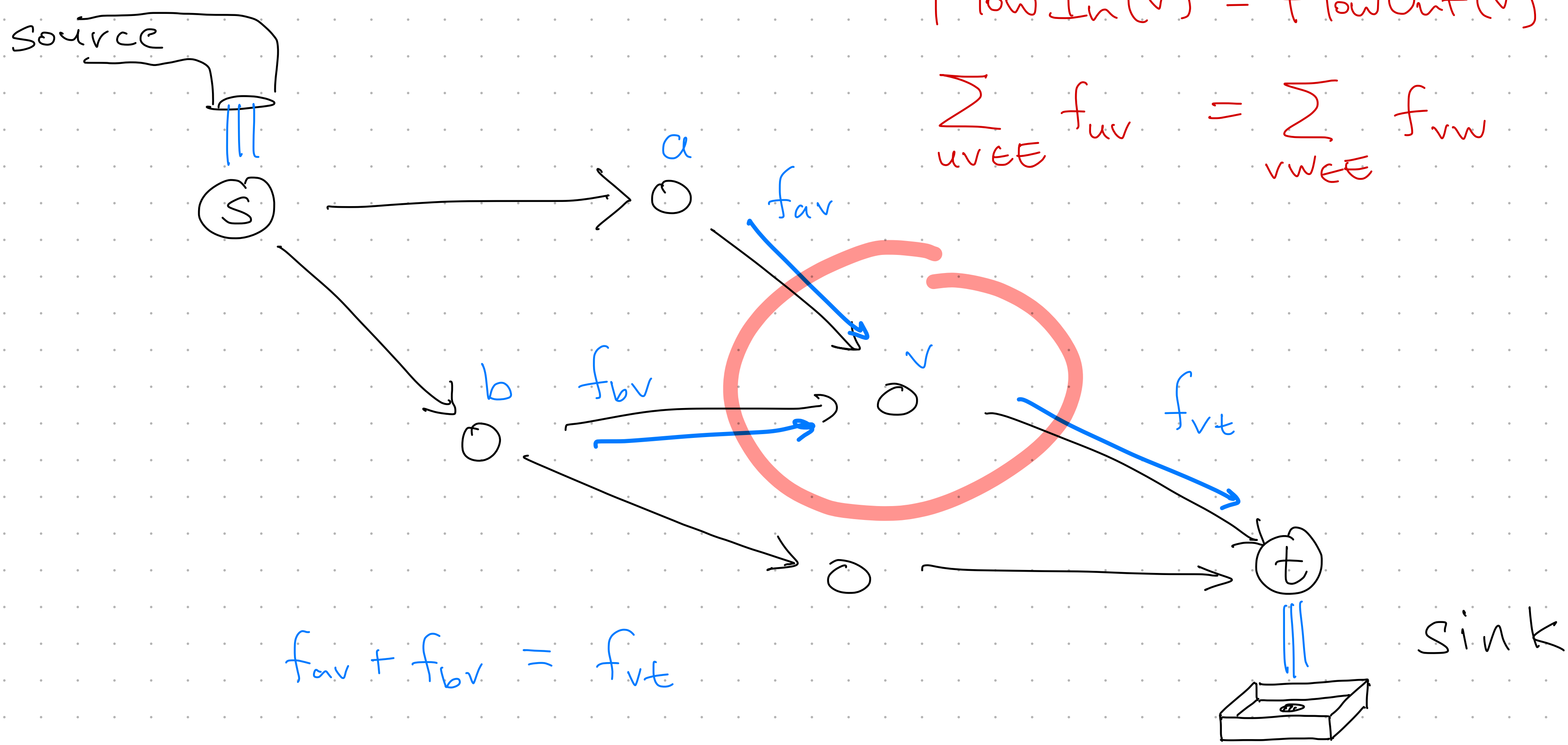
  to each edge.

Vertex conservation constraints

$$\forall v \in V \setminus \{s,t\}$$

$$\text{FlowIn}(v) = \text{FlowOut}(v)$$

$$\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw}$$

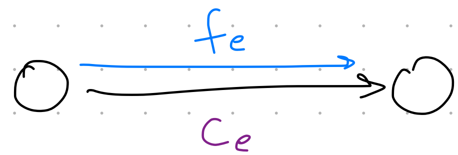


$$f_{av} + f_{bv} = f_{vt}$$

Network Flow Problem

* Given Flow Network $G, s, t, c: E \to \mathbb{R}^+$
* Find Flow $f: E \to \mathbb{R}^+$ subject to

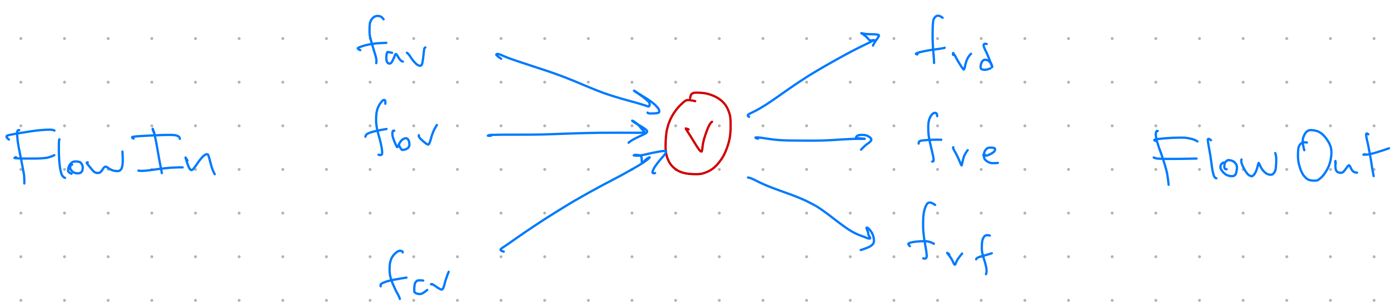
Capacity Constraints



$$0 \le f_e \le c_e \qquad \forall e \in E$$

Conservation Constraints

$$\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw} \qquad \forall v \in V \setminus \{s,t\}$$



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$$\circ \xrightarrow[c_e]{f_e} \circ$$

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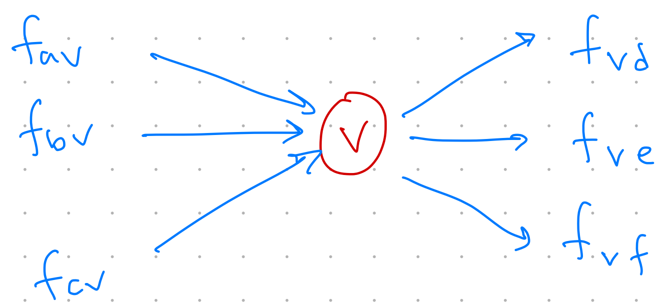
Flow In



Flow Out

Max Flow. Find Flow f^* that maximizes FlowOut (s)

Announcements

* Welcome Back — No Homework this week!

* Grading Ongoing
  - ↳ Prelim 1
  - ↳ HW2

Network Flow Problem

* Given Flow Network $G, s, t, c: E \to \mathbb{R}^+$
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Capacity Constraints $\quad 0 \le f_e \le c_e \quad \forall e \in E$

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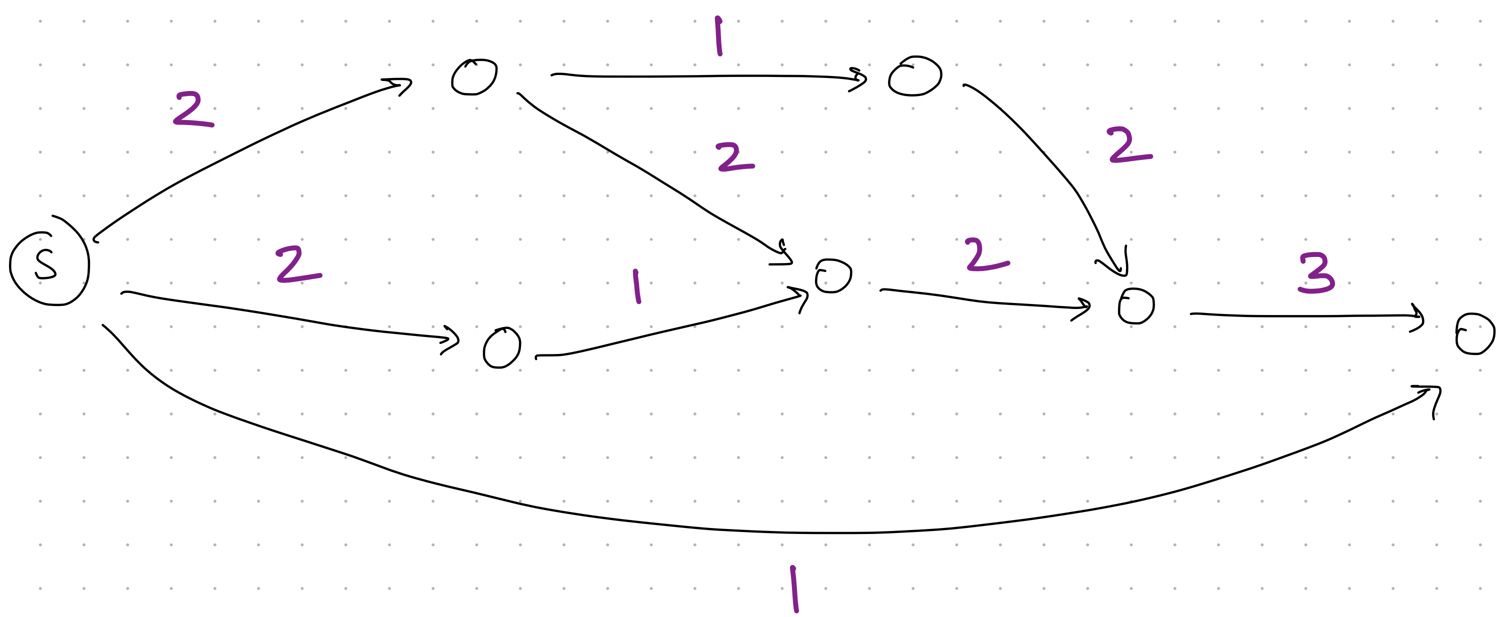

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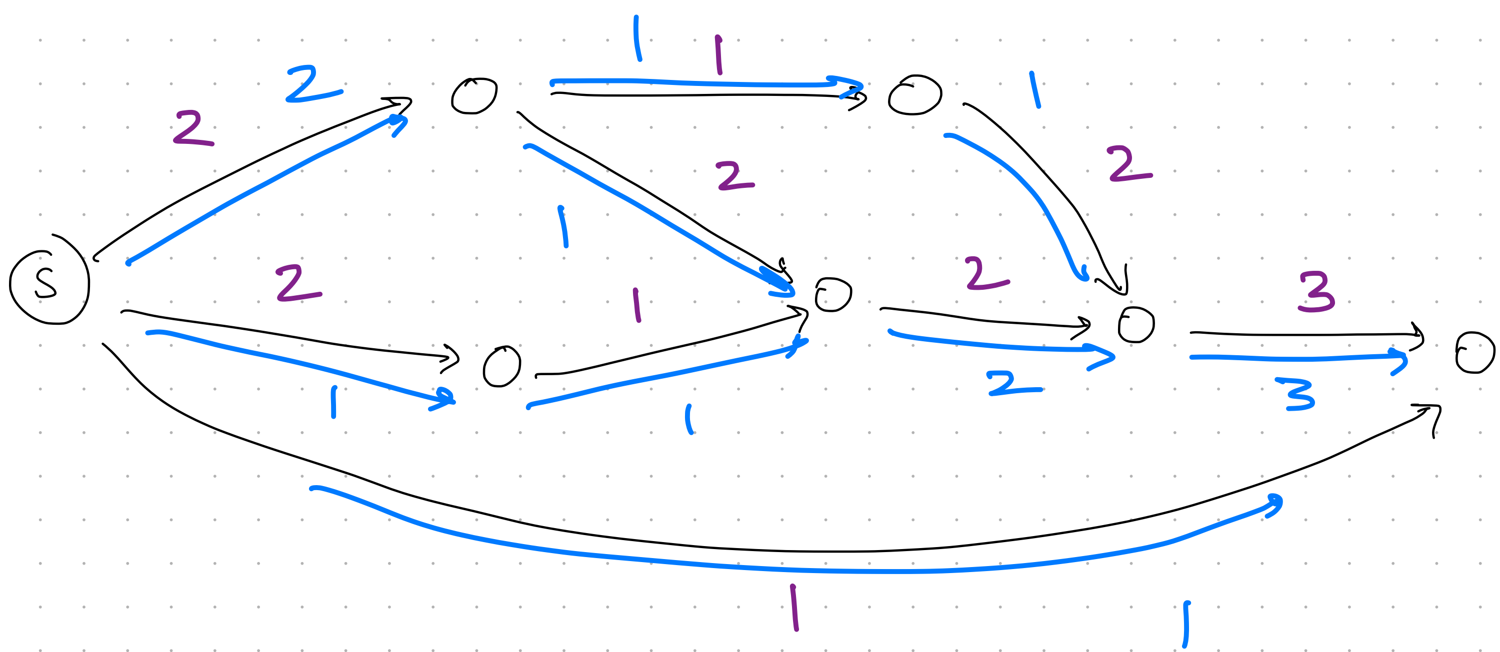

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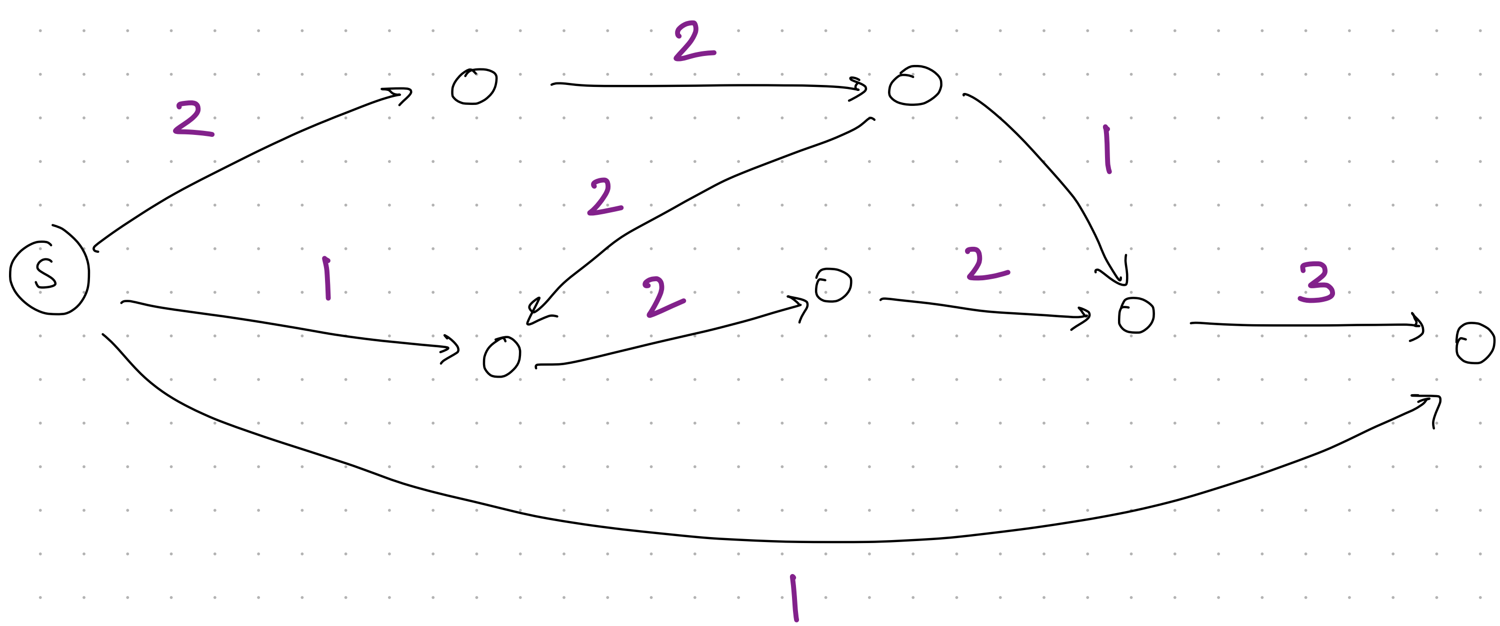

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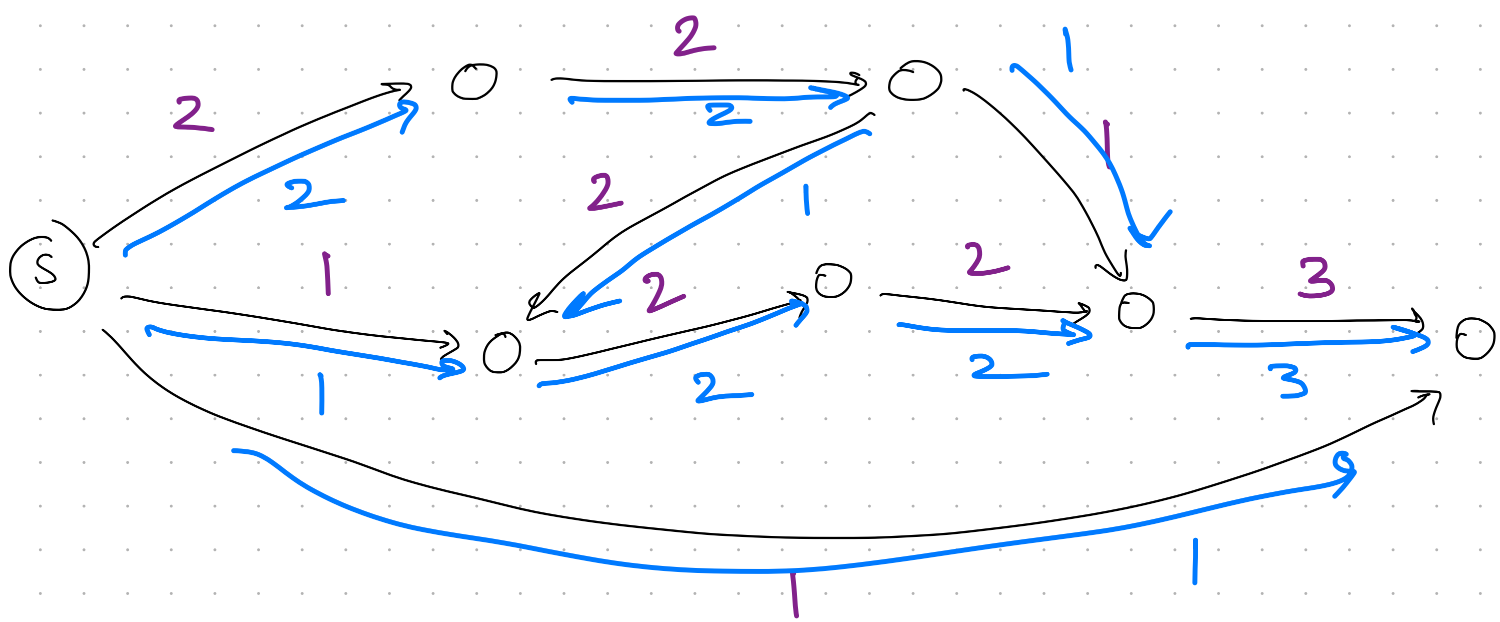

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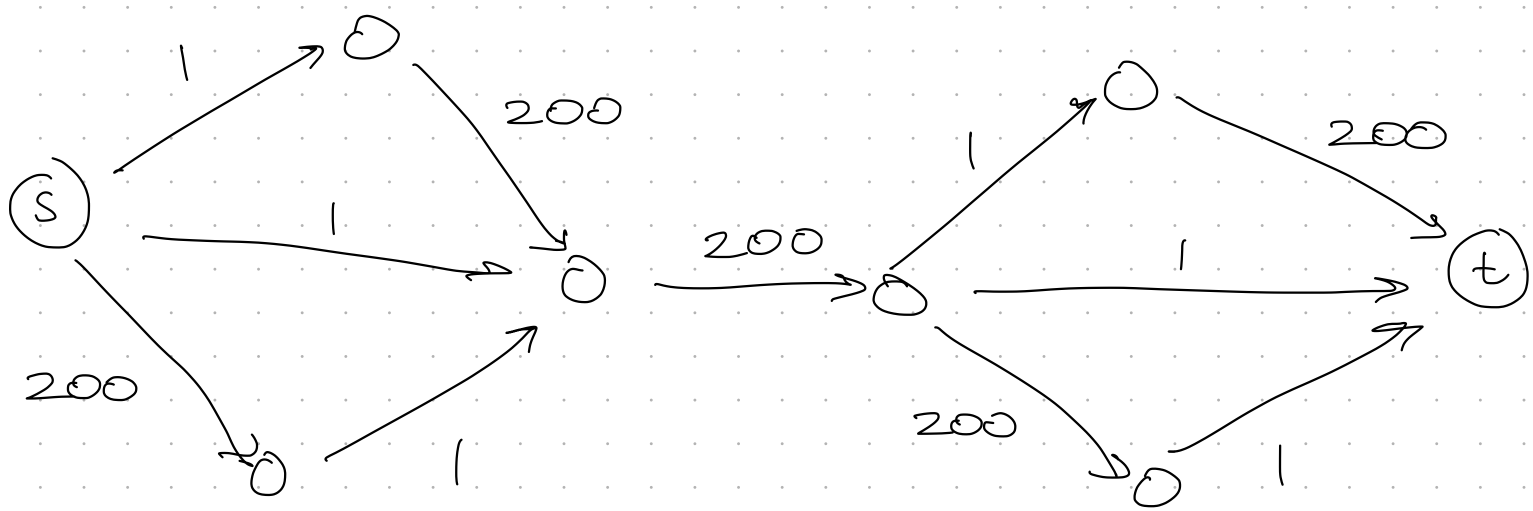

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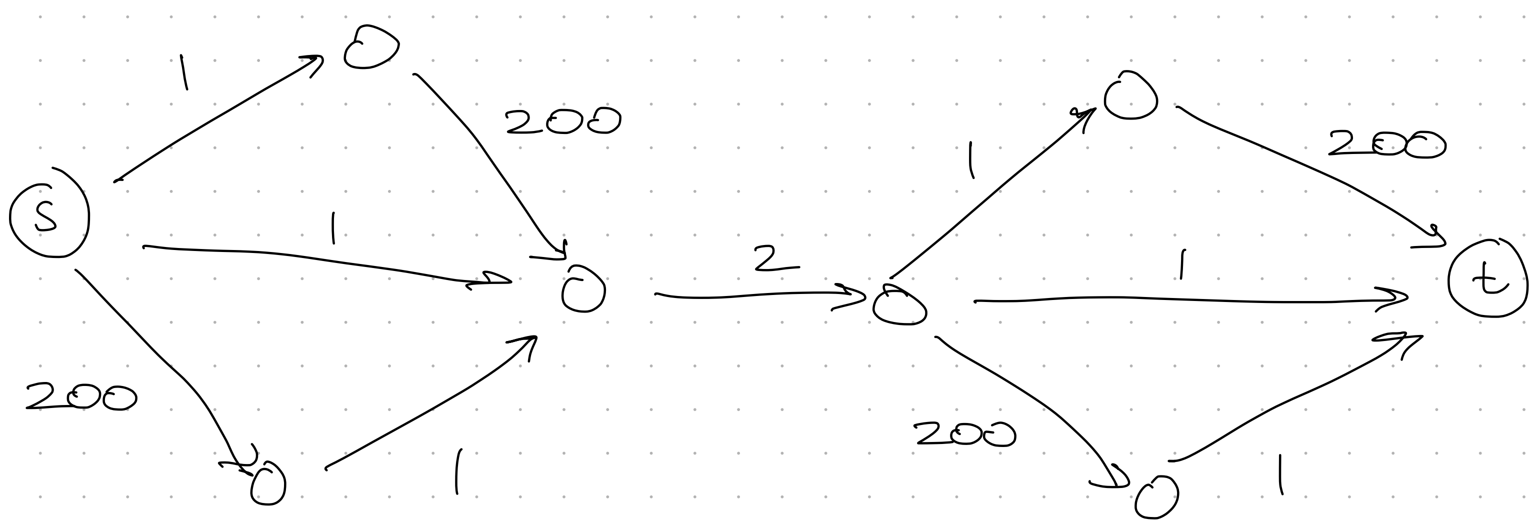

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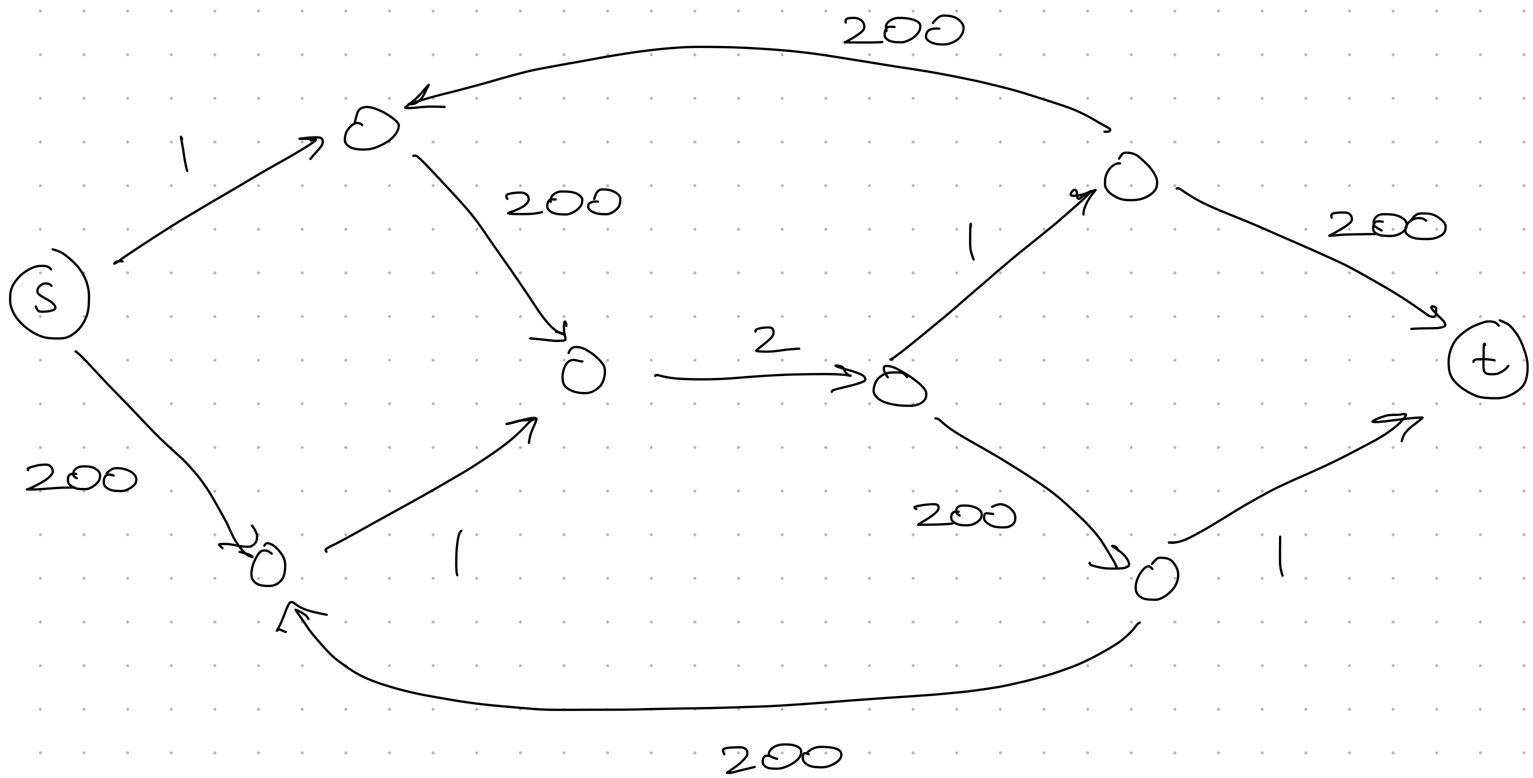

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Max Flow.

↳ Fundamental Problem

↳ Many problems can be phrased as MF.

i.e. Many problems Reduce to MF.

Maximum Bipartite Matching

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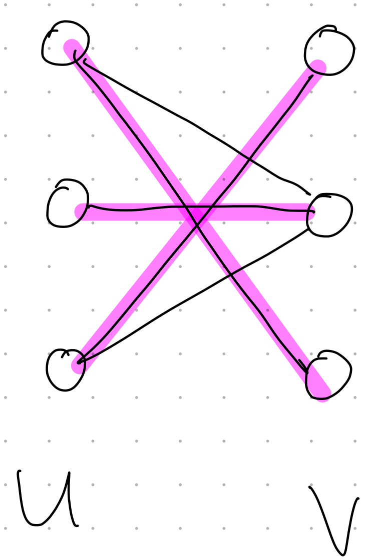
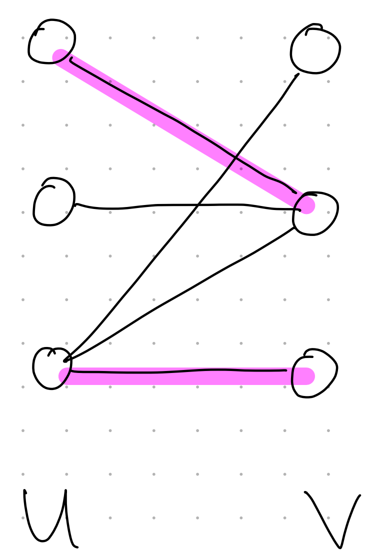
Collection of edges s.t. every vertex is in at most one edge.





Reducing Max Bipartite Matching to Max Flow

* Reduction is an <u>algorithm</u> R.

$$\text{Bipartite Matching instances} \xrightarrow{R} \text{MaxFlow instances}$$

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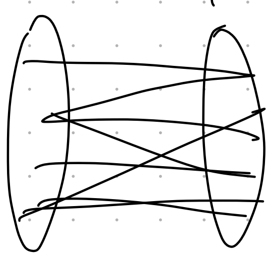
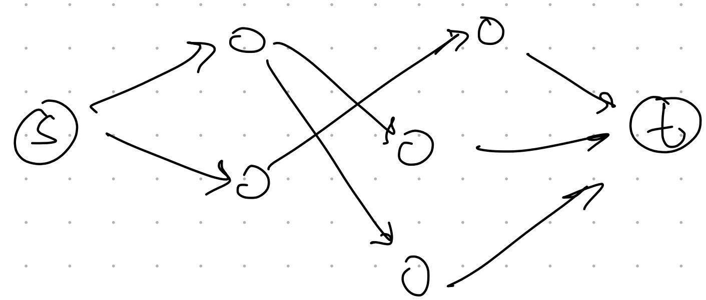
Bipartite Graph
$B = (U, V, E)$



B

Flow Network
$R(B) = (G, s, t, c)$



$R(B)$

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$$B \qquad\qquad R(B)$$

* Requirements.

$$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$$

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* <u>Requirements</u>.

($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$

then max flow in $R(B)$ is at least k.

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* <u>Requirements</u>.

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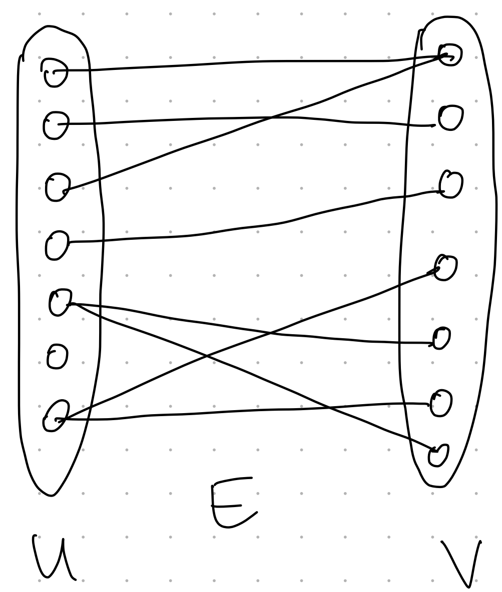
($\Leftarrow$) If $R(B)$ has max flow at least k,
then B has a matching M s.t. $|M| \geq k$.

Reduction

On input $B = (U, V, E)$

Construct G on vertices $U \cup V \cup \{s, t\}$

B



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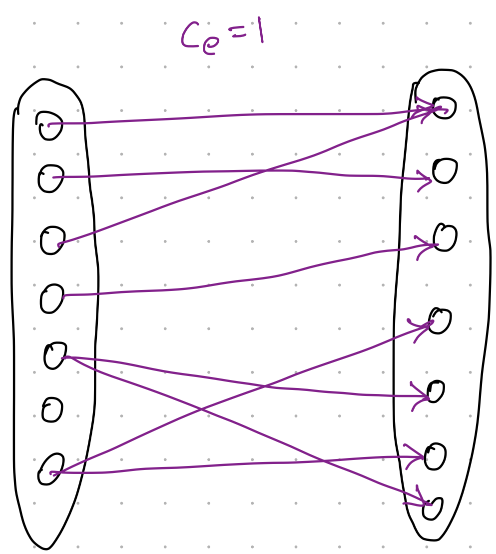
On input $B = (U, V, E)$

Construct G on vertices $U \cup V \cup \{s, t\}$

↳ For every $(u,v) \in E$, add directed edge to G w/ capacity $c_{uv} = 1$.



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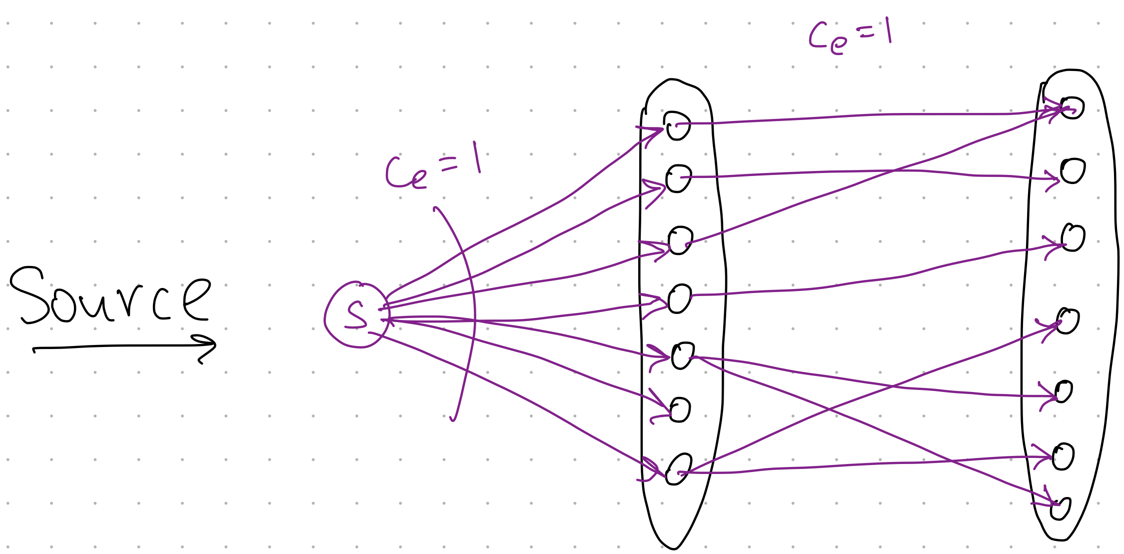
↳ For every $u \in U$, add (s,u) to G w/ cap $c_{su} = 1$



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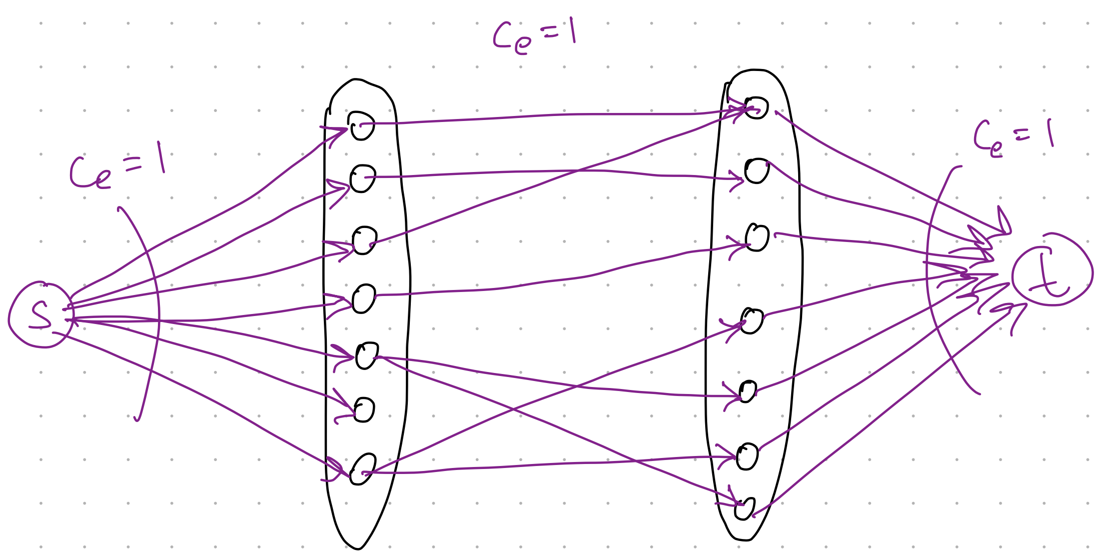
$R(B)$



sink →

Max Bipartite Match Algo

On input $B = (U, V, E)$

Run Reduction to Max Flow $R(B)$

Compute $k \leftarrow \text{MaxFlow}(R(B))$

Return k. // Returns cardinality of Max Matching in B.

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Running Time.

$$\begin{cases} \text{Time of Reduction} \rightarrow O(|U|+|V|+|E|) \\ \text{Time to Solve Max Flow} \end{cases}$$

$$\rightarrow MF(\underbrace{O(n)}_{\text{nodes}}, \underbrace{O(m+n)}_{\text{edges}})$$

Proof of Correctness.

* Need to show

$$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$$

* What can we leverage?

Proof of Correctness.

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$$\boxed{\text{Max Matching in } B = k} \iff \boxed{\text{Max Flow in } R(B) = k}$$

* What can we leverage?
  - ↳ Defn of matching → $\forall u \in U$, $v \in V$ in at most one pair in M
  - ↳ Flow constraints

    CAPACITY → $0 \leq f_e \leq c_e \quad \forall e \in E$

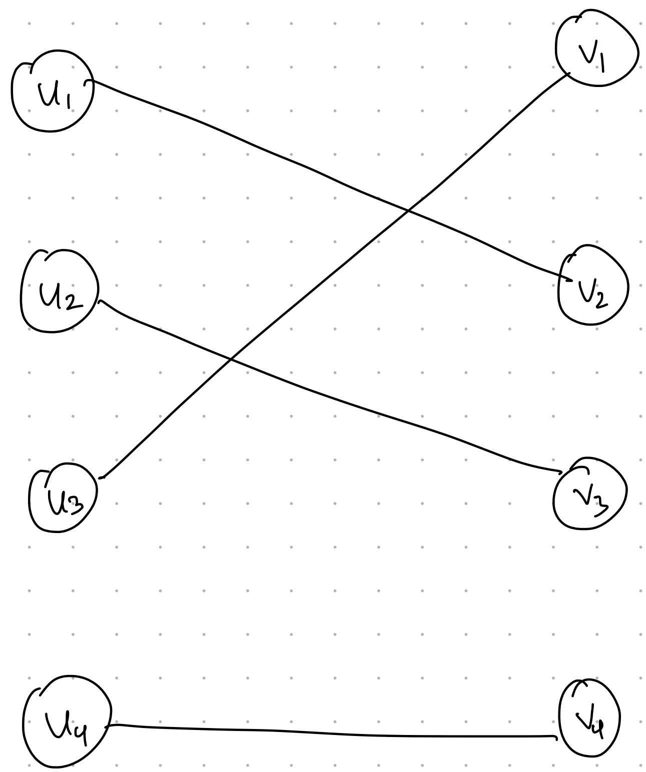
    CONSERVATION → $\text{FlowIn}(v) = \text{FlowOut}(v) \quad \forall v \in V$
  - ↳ Properties of Reduction R

Proof of Correctness.

To show.

($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$,

then max flow in $R(B) \geq k$.



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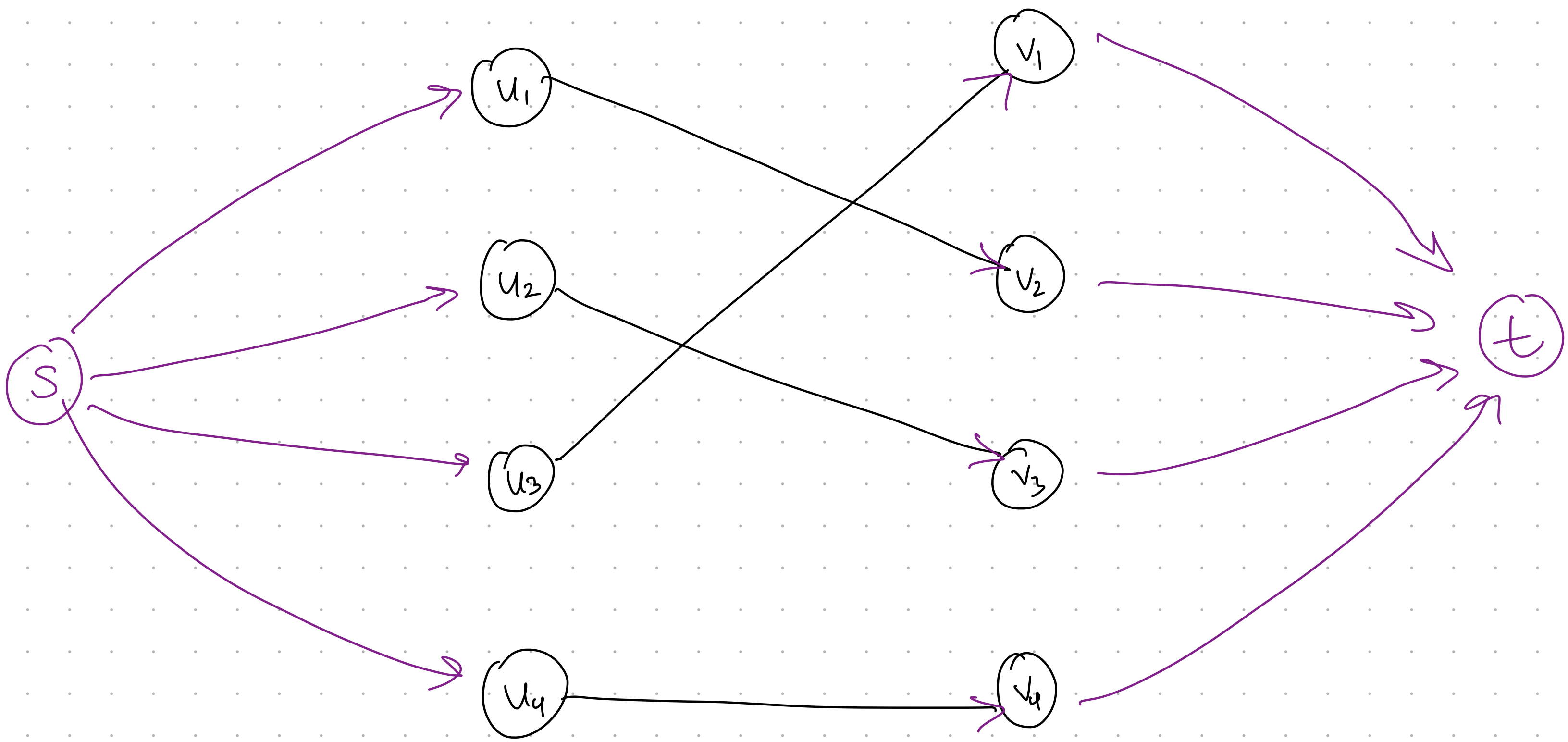

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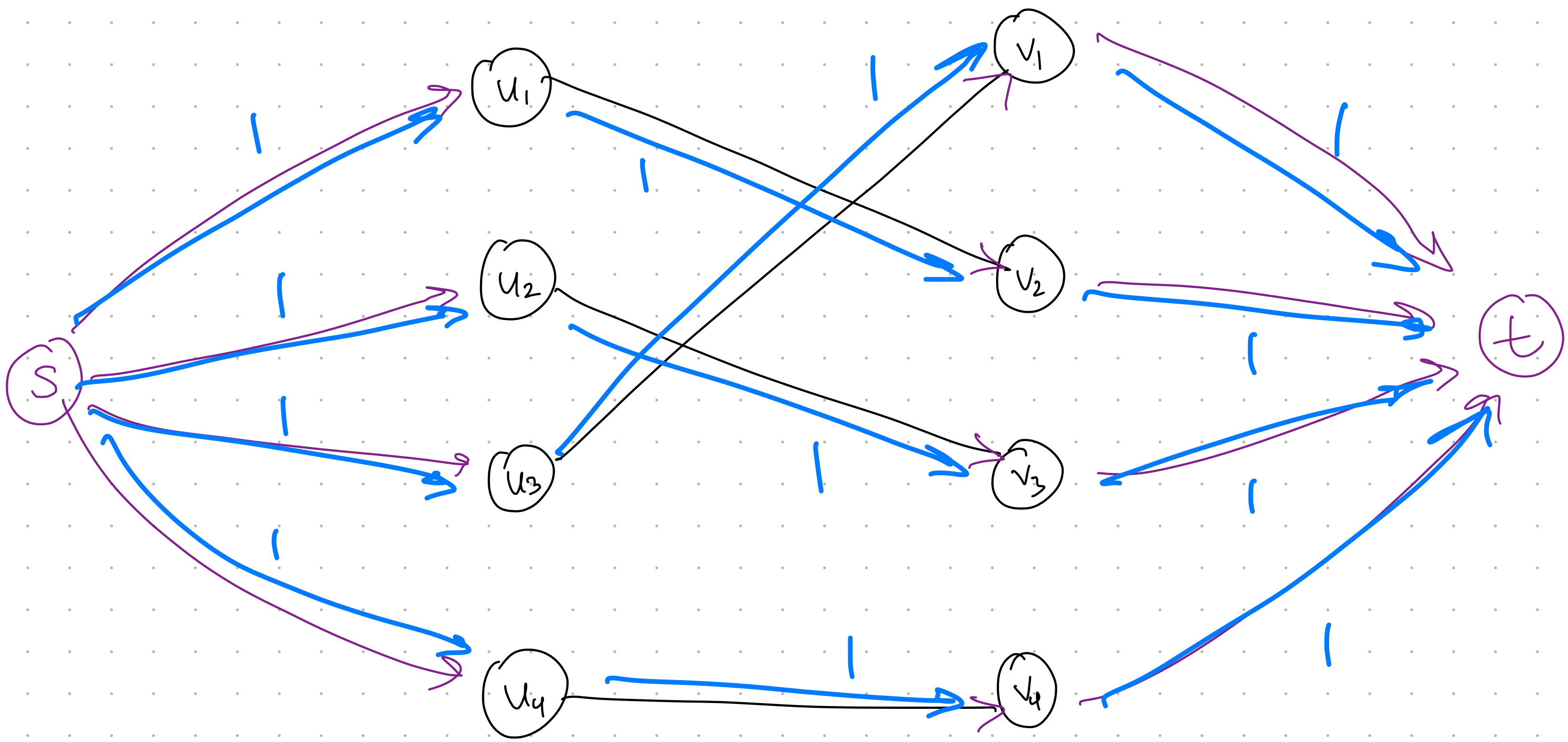


($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$,
then max flow in $R(B) \geq k$.

Consider routing 1 unit of flow along each matched edge $(u,v) \in M$

For all $(u,v) \in M$

$$f_{su} = f_{uv} = f_{vt} = 1$$

For all other edges $f_e = 0$.

We show that f is a feasible flow that routes k units

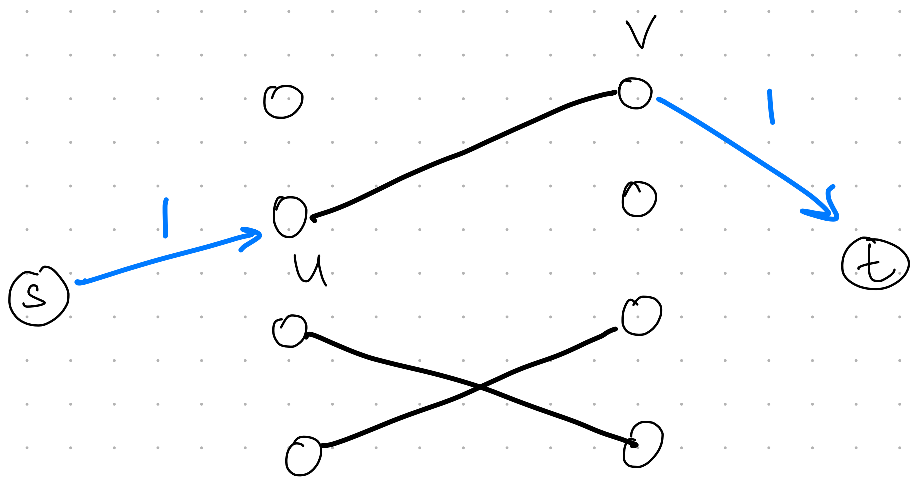
($\Rightarrow$) If B has a matching M s.t. $|M| \geq k$,
then max flow in $R(B) \geq k$.

Consider routing 1 unit of flow along each matched edge $(u,v) \in M$

By defn. of matching,

$u \in U$ and $v \in V$ involved in at most 1 matched edge

$\Rightarrow \forall u \in U$ FlowIn$(u) \leq 1$ & $\forall v \in V$ FlowOut$(v) \leq 1$.



$\Rightarrow$ FlowOut$(s) = k$.

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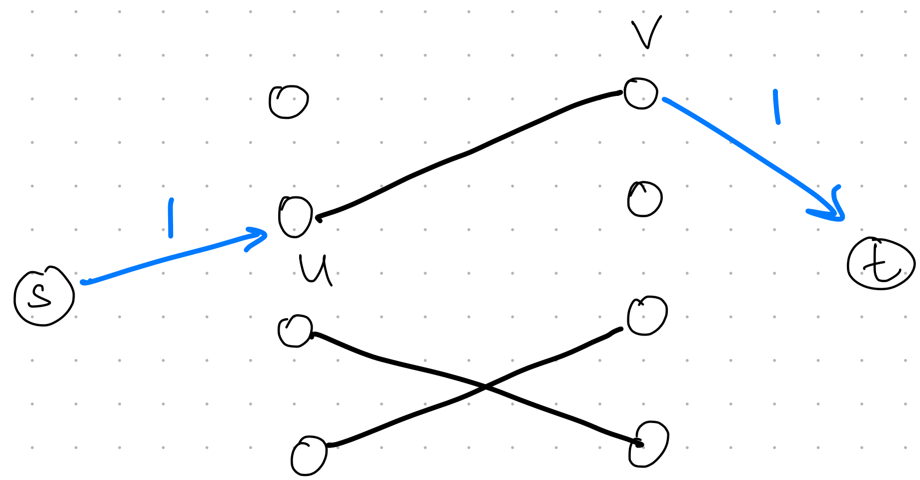

$\Rightarrow$ FlowOut$(s) = k$.

$\Rightarrow$ Capacity constraints on $s \rightarrow u$ and $v \rightarrow t$ OK ✓

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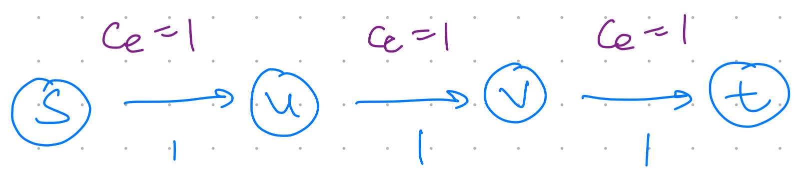

Each edge exists in $R(B)$ w/ capacity 1.

& Flow is conserved.

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$\Rightarrow \forall u \in U$ $\text{FlowIn}(u) \leq 1$ & $\forall v \in V$ $\text{FlowOut}(v) \leq 1$.

$$s \xrightarrow[1]{c_e = 1} u \xrightarrow[1]{c_e = 1} v \xrightarrow[1]{c_e = 1} t$$

Each edge exists in $R(B)$ w/ capacity 1. ✓

& Flow is conserved. ✓

CAPACITY & CONSERVATION ok $\Rightarrow$ f is feasible!

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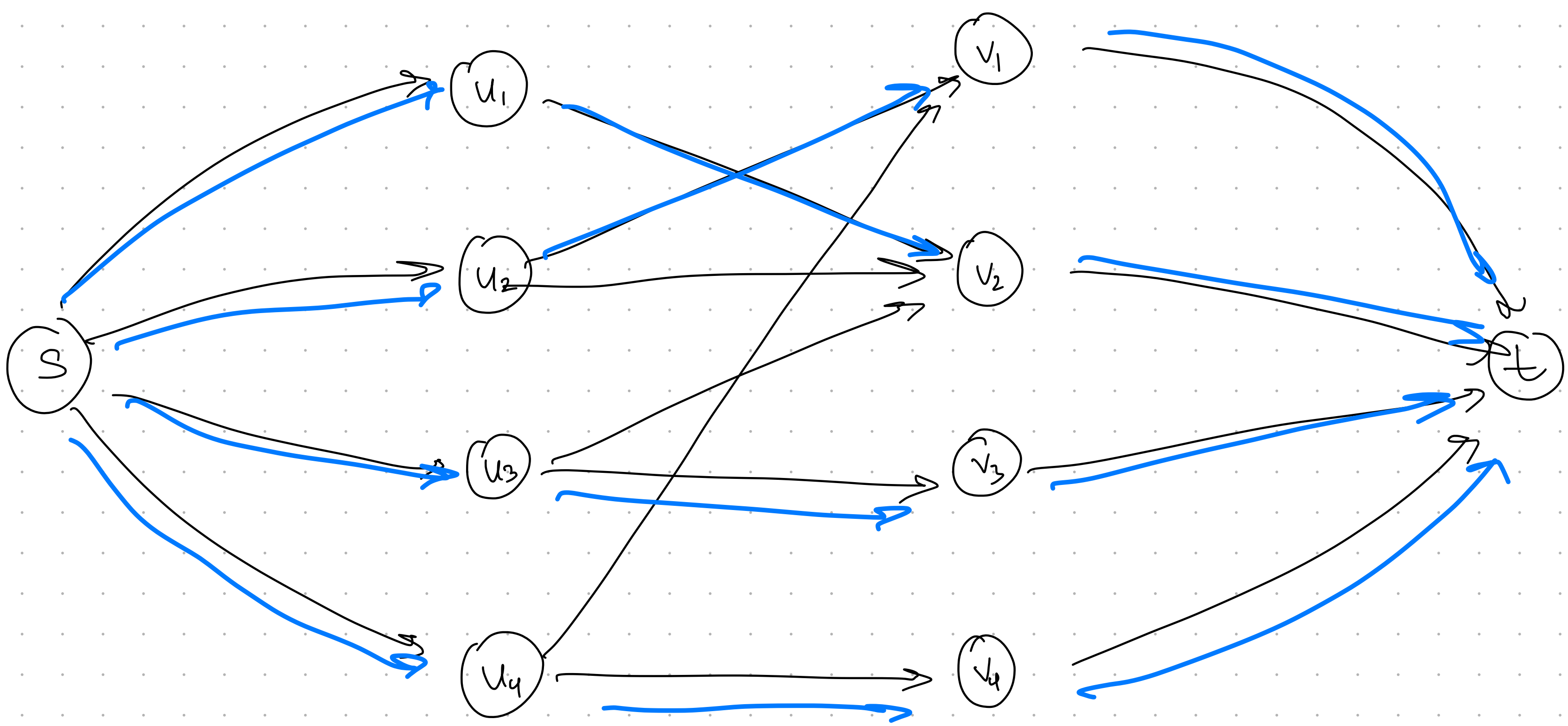

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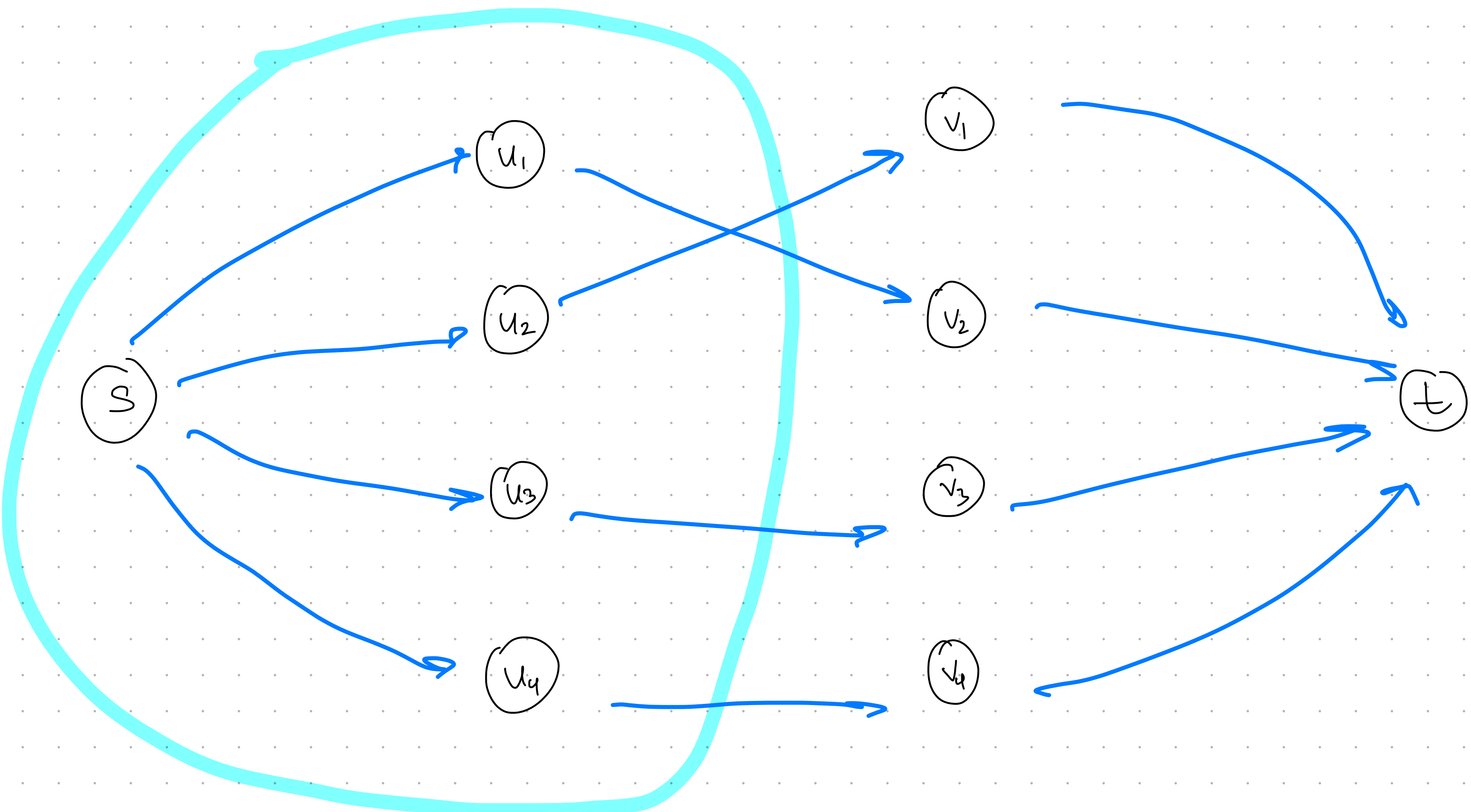

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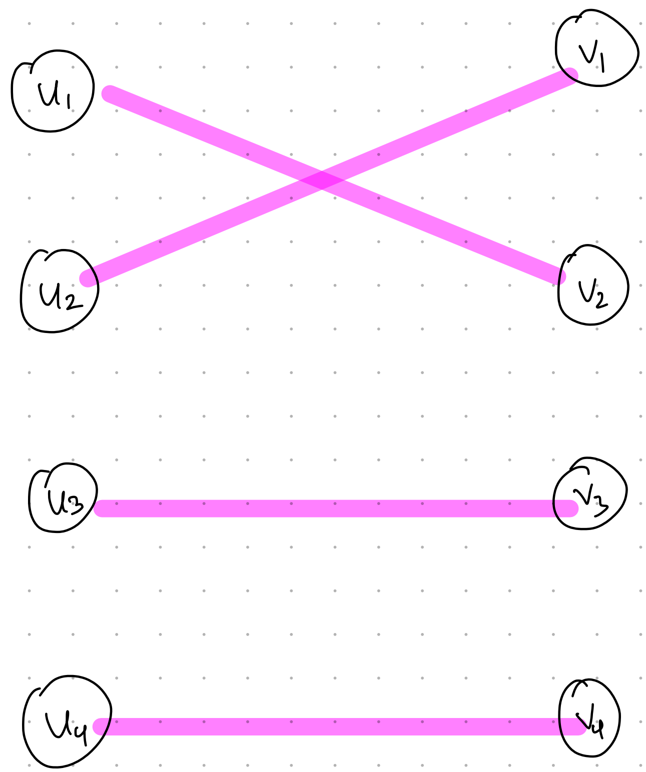


($\Leftarrow$) If max flow in $R(B) \geq k$,

then max matching in $B \geq k$.

<u>Fact</u>. If all capacities are integer, $\exists$ a max flow s.t. $f_e \in \mathbb{N}$ $\forall e \in G$

Consider such an integral max flow f.

$\Rightarrow$ By capacities $\Rightarrow$ $f_e \in \{0,1\}$ $\forall e \in G$

$\hookrightarrow$ Let $M = \{(u,v) : u \in U,\ v \in V \text{ and } f_{uv} = 1\}$

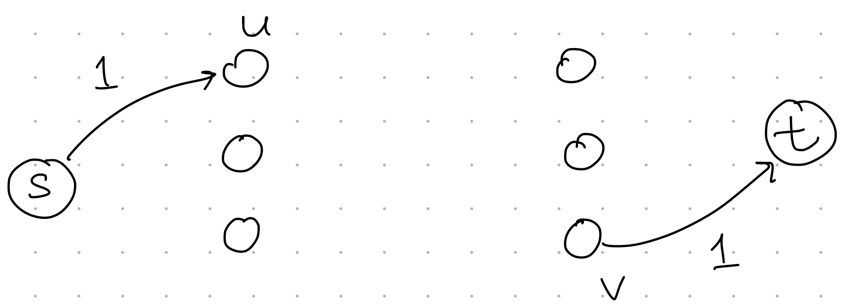
We prove M is a matching $|M| \geq k$.

Let $M = \{ (u,v) : u \in U, v \in V \text{ and } f_{uv} = 1 \}$

Claim. M is a matching.

Pf. By construction in Reduction R

* every $u \in U$ has exactly one in-edge of cap $c_{su} = 1$
* every $v \in V$ has exactly one out-edge of cap $c_{vt} = 1$



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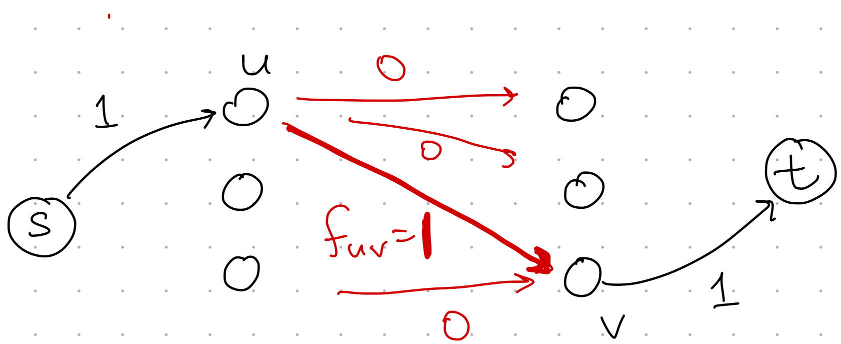

By conservation constraints.

$\Rightarrow$ At most 1 unit of flow out of every $u \in U$

$\Rightarrow$ At most 1 unit of flow into every $v \in V$.

Let $M = \{ (u,v) : u \in U, v \in V$ and $f_{uv} = 1 \}$

<u>Claim.</u> M is a matching.

<u>Pf.</u> By construction in Reduction R

* every $u \in U$ has <u>exactly</u> one in-edge of cap $c_{su} = 1$
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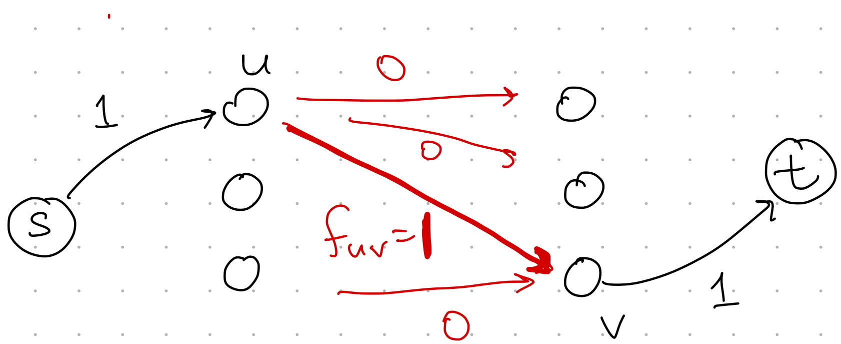


<u>By conservation constraints.</u>

$\Rightarrow$ At most 1 unit of flow out of every $u \in U$

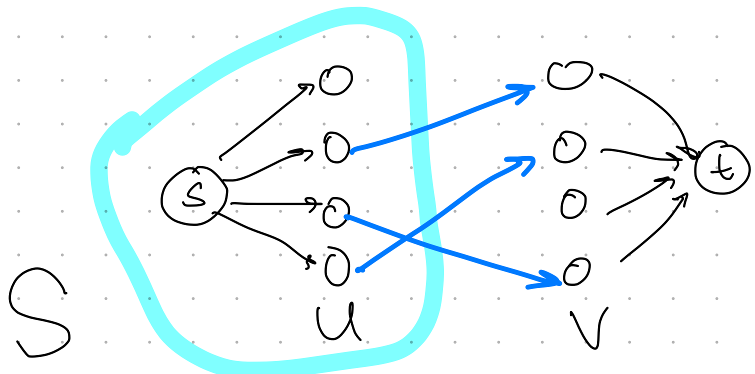
$\Rightarrow$ At most 1 unit of flow into every $v \in V$.

$\Rightarrow$ Every $u \in U$ and $v \in V$ involved in <u>at most</u> one edge in M ✓

Let $M = \{ (u,v) : u \in U, v \in V \text{ and } f_{uv} = 1 \}$

<u>Claim.</u> $|M| = k$

<u>Pf.</u> Consider the cut in G, $S = \{s\} \cup U$.



Let $M = \{(u,v) : u \in U, v \in V \text{ and } f_{uv} = 1\}$

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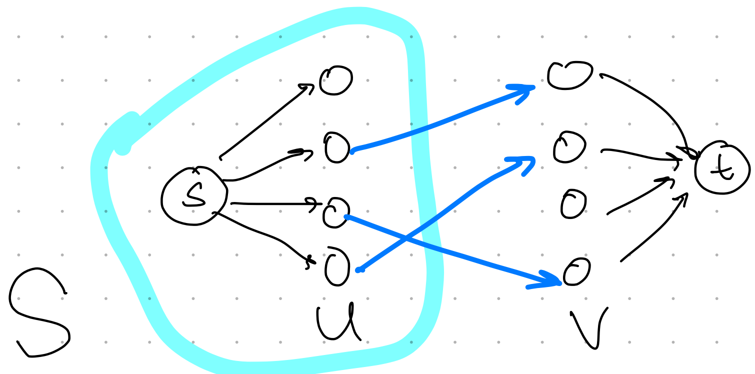

Note. No edges are oriented into S

$\Rightarrow$ FlowOut(s) $= k$ = Flow out of S

= Flow from $u \in U$ to $v \in V$.

Let $M = \{ (u,v) : u \in U, v \in V \text{ and } f_{uv} = 1 \}$

<u>Claim.</u> $|M| = k$

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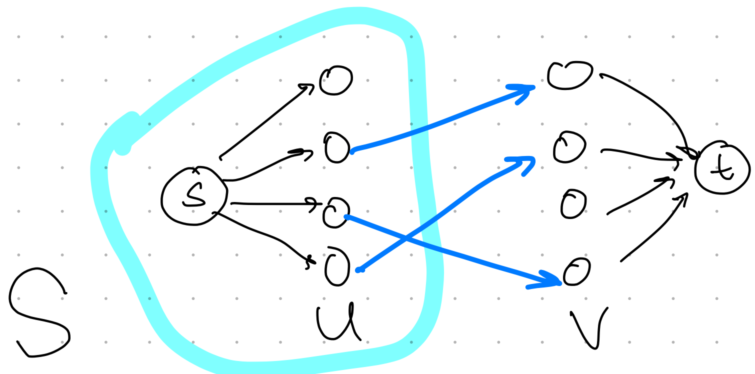

<u>Note.</u> No edges are oriented into S

$\Longrightarrow$ FlowOut$(s) = k =$ Flow out of S

$=$ Flow from $u \in U$ to $v \in V$.

By integrality of $f_e \in \{0,1\} \Longrightarrow k$ edges have non-zero flow from U to V.

$\Longrightarrow |M| = k$. ✓