5 February 2025	Dynamic Programming III
· · · · · · · · · · · · · · · · · · ·	Analysis of Edit Dist
Parameter a production of the term of te	
x Review Edit I)istance Problem
* Announcements	· · · · · · · · · · · · · · · · · · ·
* Proofs of Align	ment
* Extensions	
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Edit Distance
Given two strings S, T Find minimum sequence of edits
to furn S into T
S = C R Y P T O G R A P H Y
T = ENCRYPTING

	Distance
	two striles S, T
•	minimum sequence of edits
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· · · · · · · · · ·	
	CRYPTOGRAPHY
	ENCRYPT ING
	Addition to T (cost X)
· · · · · · · · · ·	e.g. = 1

Edit Distance	· · ·
Given two strags S, T	
Find minimum sequence of edits to turn S into T	
Deletion from S	· · ·
S = C R Y P T O G R A P H X	· · ·
TENCRYPTING	· · ·
(cost)	· · ·

Edit Distance
Given two strings S, T
Find minimum sequence of edits to turn S into T
S = C R Y P T O G R A P H Y
T= ENCRYPTONG
Change Siz -> Tj
$(1 \circ S t) \land S \circ T \circ S \circ S \circ T \circ S \circ S \circ S \circ S \circ S \circ$

Edit Distance
Given two strings S, T Find minimum sequence of edits
to furn S into T
S = C R Y P T O G R A P H Y
T = ENCRYPTING

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Find	no strigs S, T nimum sequence of edits S into T	· · · · · · ·
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	$C_{1} = C_{1} = C_{1$	
	CRYPTOGXARMX	 . .

Announcements
XHWL Due XHWL Out Today
* Prehim #1 L. Next Thurs. 13 Feb 7:30-9p. L. Closed Note, Closed Internet.
★ Review Session Lo Saturday Recitation Lo Next Tues. II Feb 7-9p Gates G01

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Edit Distance Given two strigs S, T Find minimum sequence of edits to furn S into T CRYPTOGRAPHY ENCRYPTING What do we know about the optimal Solution?

Edit Distance Given two strags S, T Find minimum sequence of edits to furn S into T CRYPTOGRAPHY ENCRYPTING Fact 1. In every sequence of edits, each character in S is either * Deleted * Changed/Aligned into T

Edit Distance Given two strigs S, T Find minimum sequence of edits to furn S into T CRYPTOGRAPHY ENCROPTING In every sequence of edits, Fact 2 each character in T is either * Added * Changed/Aligned from S

Fact 1 + Fact 2 => Exhaustice Case Analysis	2
CaseS	• •
* Sn is deleted	• •
* Trn is added	• •
* At least one of them is aligned/changed.	• •
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=> Exhaustice Case Analysis Fact 1 + Fact 2 Sn deleted C = R = Y = P = T = O =ENCORY edit distance between S[I:n-1] and T edit distance between S and T

=> Exhaustice Case Analysis Fact 1 + Fact 2 Tru is added C. R. Y. P. T. O.G. ENCCY edit distance between S and T [1:m-1] edit distance between S and T

Fact 1 + Fact 2 => Exhaustice Case	Analysis
At least one aligned?	
CRYPTOG What happened? ENCRY	
C : R : Y : P : T : O : G :	· · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · · ·
ENCRYE what happened?	
	· · · · · · · · · · ·

Fact 1 + Fact 2 => Exhaustice Case Analysis At least one aligned? Sn aligned w/ Tm CRYPTOG ENCRY edit distance between S[1:n-1] and T[1:m-1] edit distance between S and T ShTm

=> Exhaustice Case Analysis Fact 1 + Fact 2 (Cases) * Sn is deleted * The is added * Sn aliqued to Tm DP Recurrence Edit distance between S[1:n-1] and T 7 2 + Edit Distance Edit distance between S and T[1!m-1] = win between S and Shint Edit distance Shint between S[1:1-1] and J [1:m-1]

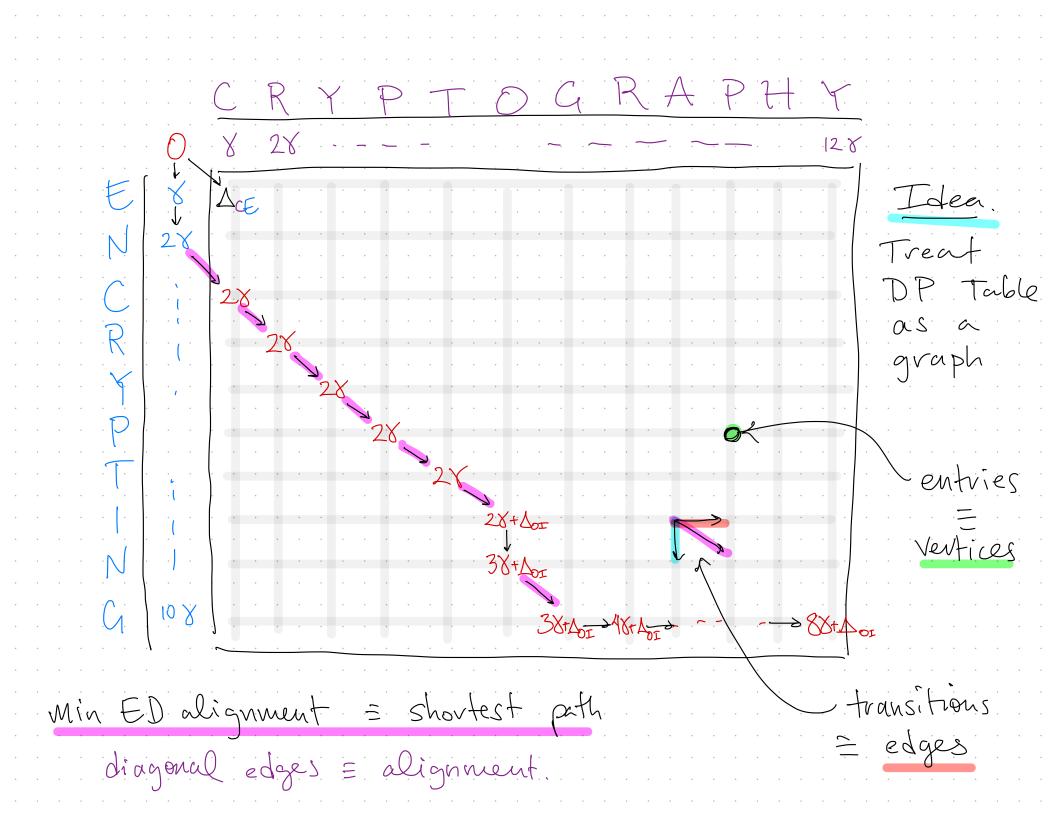
Correctness. Consider the recurrence ED gier as $ED(n,m) = min \{ ED(n-1,m-1) + A_{snTm}, \}$ ED(n,m) + 3, Tm = 1, Tmwhere ED(0,0) = 0, ED(1,0) = Y, ED(0,1) = YTheorem For any n, m EN. suppose 151=n, 171=m. Edit Distance between S and = ED(M, M)

Edit Distance Algorithm ED(o, o) = O $(i,0) = i \cdot Y$ $ED(O,j) = j \cdot Y$ For 2=1->n $For j = 1 \longrightarrow M$ $\left\{ \begin{array}{l} ED(i,j) = \min \left\{ ED(i-1,j-1), ED(i-1,j), ED(i,j-1) \right\} \\ + \Delta_{s_i T_j} + \gamma + \gamma + \gamma + \gamma \end{array} \right\}$ refurn ED(n,m)

TOGERAPHY 28 12.8 28 Each entru requires 3 probes ED(i-1,j-1) ED(i,j-1) $ED(i,j) \times ED(i,j)$ into prior entries. 10 X D(nm)

Edit Distance Algorithm ED(o, o) = OTheorem. The Edit Distance $ED(i,0) = i \cdot Y$ DP Algorithm runs in $ED(0,j) = j \cdot Y$ a a a (D) (m, m). A fine For 2=1-2n For $j = 1 \longrightarrow M$ $\left\{ \begin{array}{l} ED(i,j) = \min \left\{ ED(i-1,j-1), ED(i-1,j), ED(i,j-1) \right\} \\ + \Delta_{s_i T_j} + \gamma + \gamma + \gamma + \gamma \end{array} \right\}$ refurn ED(n,m)

Edit Distance = Sequence Alignment CRYPTO GRAPHY ENCRYPTING An alignment is non-crossing matching. ED Algorithm computed the distance Can we compute the alignment?



Theorem. For any n, m EN. suppose (SI=n, (T)=m shortest path Edit Distance between S and T $(O, O) \rightarrow (O, O)$ in ED. Graph (0, 0)PTOGRAPHY CRYPTO GRAPH' ENCRYPTING $\frac{2X+\Delta_{0T}}{3X+\Delta_{0T}} \xrightarrow{AX+\Delta_{0T}} \xrightarrow{AX+$ D'agonal edges. Vio

Fact 3 Eve in	the ED Graph	D) to (n,m) uses one of
· · · · · · · · · · · · · · · · · · ·	hvee edges:	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$		$ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1$
$ \begin{array}{c} & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & $	\longrightarrow (α, α) (α, α) (α, α)	

Theorem. For any n, m EN. suppose (SI=n,)TI=m. shortest path Edit Distance between S and T $(0, 0, 0) \rightarrow (0, 0, 0)$ in ED Graph Proof By induction on n+M. * Two base cases (direct analysis) (based on ED Remirrence) & Fact 3 Inductive Step

suppose (SI=n, (T)=m. For any n, MEN. shortest path Edit Distance between S and T $= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \xrightarrow{\alpha} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$ in ED Graph n + m = 0and n+m=1Base Cases When n+m=0, |S|=|T|=0so both strings equal the empty string. $\int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}$

For any n, m E N suppose (SI=n, IT)=m shortest path Edit Distance between S and $= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \xrightarrow{1}_{n} \xrightarrow{1}_{n} \xrightarrow{1}_{n} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ in ED Graph n + m = Oand n+m=1Base Cases When n+m=0, |S|=|T|=0so both strings equal the empty string. shortest path Edit Distance between "" and " $\left(\left(\bigcirc 1, \left(\odot 1, \left(\bigcirc 1, \left(\odot 1, \left$ () (= in ED Graph

For any n, m EN. suppose (SI=n, (T)=m. shortest path Edit Distance between S and T $= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \xrightarrow{\alpha} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$ in ED Graph Base Cases n+m = 0, and n+m=1 When h+m=1, then |S|=1 and ITI=0. (or vice versa) S can be turned to T via one deletion Edit Distance between "C" and "" $r \equiv r X$ r

For any n, m EN. suppose (SI=n, IT)=m. shortest path Edit Distance between S and T $= \left(\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{} \left(\begin{array}{c} 0 \end{array} \right) \xrightarrow{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{} \left(\begin{array}{c} 0 \end{array} \end{array}$ in ED Graph N + M = 0and n+m=1Base Cases * The vertex (1,0) has 1 in-edge * from (0,0) w/ length X shortest path $= \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \left(\begin{array}{c} 0 \end{array} \right) \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \left(\begin{array}{c} 0 \end{array}$ in ED Graph

For any n, m EN. suppose (SI=n, IT)=m. shortest path Edit Distance between S and T $= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \xrightarrow{(1)} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$ in ED Graph Base Cases n+m = 0, and n+m=1 When n+m=1, Then |S|=1 and ITI=0. shortest path Edit Distance between "C" and "" $= \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2}$ in ED Graph (0,0)~>(0,1) similar

For any n, MEN. suppose (SI=n, IT)=m. shortest path Edit Distance between S and (10,0)in ED Graph Inductive Step It $\forall i_{ij}$ such that $i_{i+j} \leq n + m$ shortest path Edit Distance between S[1:i] and T[1:j) $(0,0,0) \longrightarrow (1,0)$ in ED Graph

suppose (SI=n, IT)=m. For any n, m EN. shortest path Edit Distance between S and $\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \begin{array}{c} 0 \\ 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{c} 0 \\ 0 \end{array}\right), \begin{array}{c} 0 \\ 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{c} 0 \\ 0 \end{array}\right), \begin{array}{c} 0 \\ 0 \end{array}\right)$ in ED Graph Inductive Step First, Recall ED Recurrence Edit distance between S[1:n-1] and T Edit Distance between S and T Edit distance between S and T[1!m-1] = Mih Shint Edit distance Shint between S[1:1-1] and T[1:m-1]

For any n, m EN. suppose |S|=n, |T|=m. shortest path Edit Distance between S and T $= \left(\left(\begin{array}{c} 1 \\ 1 \end{array}\right) \left(\begin{array}{c} 1 \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \end{array}\right) \left(\begin{array}{c} 1 \end{array}\right)$ in ED Graph Inductive Step By Fact 3, (n, m) has 3 in-edges, so shortest path is the min of in-edge wts t shortest path to heighbor. $\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} , 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix}$ $(M_{1}, M_{2} - 1)$ $\sum_{n=1}^{n} (n-1, n-1)$ Shortest path from (0,0) (n,n) \sum (n-1, m)

For any n, m E N suppose (SI=n, (T)=m shortest path Edit Distance between S and T $= \left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \begin{array}{c} 0 \\ 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{c} 0 \\ 0 \end{array}$ in ED Graph Inductive Step By Fact 3, (n, m) has 3 in-edges, so shortest path is the min of in-edge wts + shortest path to heighbor. Shortest path from $(0,0) \longrightarrow (n-1,m)$ With $X + (0,0) \longrightarrow (n,m-1)$ Shortest path from (0,0) \longrightarrow (n,m)Shortest path from $(0,0) \longrightarrow (n-1, m-1)$ Sh Tim t

For any n, m EN. suppose (SI=n, IT)=m shortest path Edit Distance between S and T $= \left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \begin{array}{c} 0 \\ 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{c} 0 \\ 0 \end{array}$ in ED Graph Inductive Step ED w/ SP for itj < n+m By IH. Replace Shortest path from $(0,0) \longrightarrow (n-1,m)$ Edit distance between S[1:n-1] and T 1 J J I & F Edit distance between S and T[1:m-1] Shortest path from $(0,0) \longrightarrow (n,m-1)$ Xr Shint between S[1:1-1] and T[1:m-1] Shortest path from (0,0) ~ (n-1, m-1) Sh Ton t ED(S, T) = 0 $SP(0,0) \sim (n,m)$ U

heorem. For any n, m EN suppose 151=n, 171=m. shortest path Edit Distance between S and T $= \left(\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right) \xrightarrow{1}{} \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \begin{array}{c} 0 \\ 0 \end{array} \right)$ in ED Graph In other words, Edit Distance reduces to single source shortest path. L> Bellman-Ford generalizes ED

Reducing the Space Complexity? Table O(n m)entries X 28 12.8 X-JACE-> 28 Each entry requires ED(i-1,j-1) ED(i,j-1)3 probes $ED(i,j) \times ED(i,j)$ into prior entries. $\left(10 \right)$

Reducing the Space Complexity? A P P G R. X 28 12.8 X- ACE 28 Each entry requires ED(i-i,j-1) ED(i,j-1)3 probes ED(i-1,j) ED(i,j) into prior entries. Ly From prev. Column from top to 10 X boffon

Linear Space ED. $Prev(j) = j \cdot \gamma$ // m-entry 1D arrays Curr(j) = 0For i = 1 - p N $Curr(0) = i \cdot Y$ For $j = 1 \longrightarrow M$ $Curr(j) = min Z Prev(j-1), Prev(j), Curr(j-D), + \Delta_{siTj} + X + X$ Prev & Curr. refurn Curr (m)

Reducing the Space Complexity? A PIT G R. X 28 12.8 X- Aces 28 Each entry requires ED(i-i,j-1) ED(i,j-1)3 probes ED(i-1,j) ED(i,j) into prior entries. Ly From prev. Column from top to 10 X pottone Prev ~ Curr

Theorem There exists an algorithm solving the Edit Distance prolder vunning in O(nm) time using O(min Zn, m3) space.