24 January 2025

Greedy Algorithms: Minimum Spanning Tree

Plan

* Minimum Spanning Tree (MST) Problem
* Announcements
* Greedy Algorithms

Minimum Spanning Tree

vertices edges weights

Given a connected, undirected, weighted graph $G = (V, E, W)$

Find a minimum spanning tree $T = (V, E' \subseteq E)$

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Find a minimum spanning tree $T = (V, E' \subseteq E)$

spanning: $\forall u, v \in V$, u and v connected in T

tree: graph with no cycles

Minimum Spanning Tree

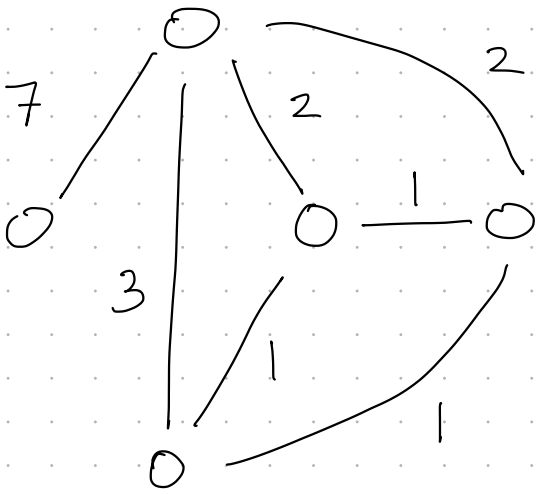
Given a connected, undirected, weighted graph $G = (V, E, w)$

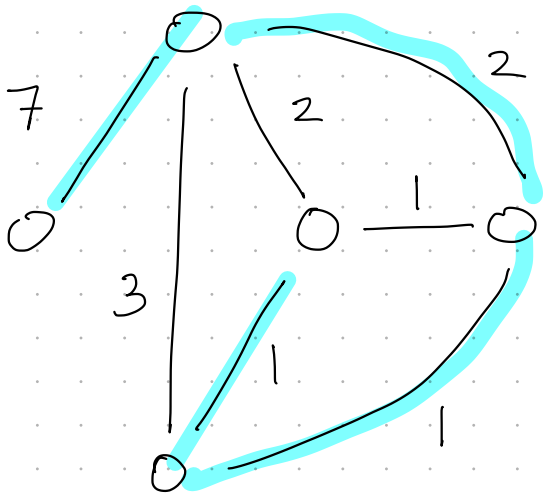
Find a minimum spanning tree $T = (V, E' \subseteq E)$

- minimum: minimize sum of edge weights.

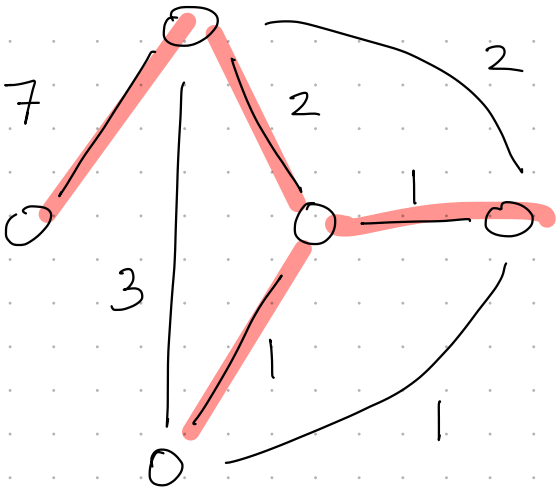
$$w_T = \sum_{e \in T} w_e$$

- spanning: $\forall u, v \in V$, u and v connected in T
- tree: graph with no cycles





$W_T = 11$



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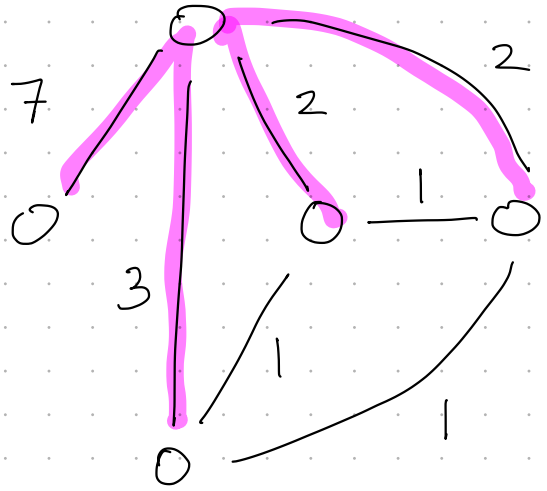
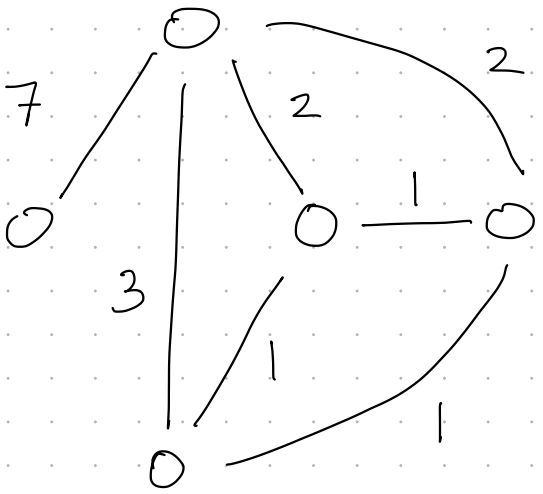


$W_T = 14$


7
2
2
1
3
1
1

Announcements.

Update to wording of HW0 Q1(d)

* Canvas published (SORRY!)
* KT added to CAMP.
* OH start this afternoon
  - ↳ Prof. Kim → Gates 203, 1-3p
  - ↳ See full schedule on course page.

Locations: Rhodes 590 ← dedicated space 4820 24/7.

Rhodes 655 ↑ large room for "peak" OH

MST Algorithms.

Given a weighted graph $G = (V, E, W)$

Return a minimum spanning tree.

Approach. Greedy Algorithms

Greedy Algorithms

Design Paradigm. Choose solution "greedily"

Myopic → local decisions that "look good"

Irrevocable → once we make a decision, we never reconsider.

Greedy Algorithms

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Advantage. Fast & Local

Running Time?

Greedy Prototype.

① Sort by "priority" $O(n \log n)$

② Iterate through elems in priority order
↳ Make decision about elem.
"constant" $O(1)$ time

$O(n)$

Greedy Algorithms

Design Paradigm. Choose solution "greedily"

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Greedy Prototype

1. Sort by "priority" ← Design of Greedy Algorithm
2. Iterate through elems in priority order
   ↳ Make decision about elem.

Greedy Algorithms

Design Paradigm. Choose solution "greedily"

Myopic → local decisions that "look good"

Irrevocable → once we make a decision, we never reconsider.

Warning. Challenging to analyze correctness

More often than not,

Greedy approaches are incorrect.

Greedy Algorithms for MST

Given graph $G = (V, E, W)$

Greedy Prototype

1. Sort by "priority"

weights

2. Iterate through elems in priority order

edges

   ↳ Make decision about elem.

Two MST Algs.

On input $G = (V, E, W)$

Step 0. Sort edges by weight

$$w_{e_1} < w_{e_2} < \cdots < w_{e_m}$$

Two MST Algs.

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Add Min

Delete Max

Two MST Algs.

On input $G = (V, E, W)$

Step 0. Sort edges by weight

$$w_{e_1} < w_{e_2} < \cdots < w_{e_m}$$

Add Min

$T = (V, \{\})$

For $i = 1 \longrightarrow m$.

  if $T \cup \{e_i\}$ does not form a cycle

    Add e_i to T.

Return T.

Delete Max

$T = (V, E)$

For $i = m \longrightarrow 1$.

  if $T \setminus \{e_i\}$ is still connected

    Remove e_i from T

Return T.

Which algorithm (if any) is correct?

Given a tree T, how do we know that

T is / is not a min spanning tree?

Which algorithm (if any) is correct?

Given a tree T, how do we know that

T is / is not a min spanning tree?

Simplifying Assumption.

Assume edge weights are distinct.

Given G, are there any edges that MUST NOT be in the MST?

(Consider cycles w/in G.)

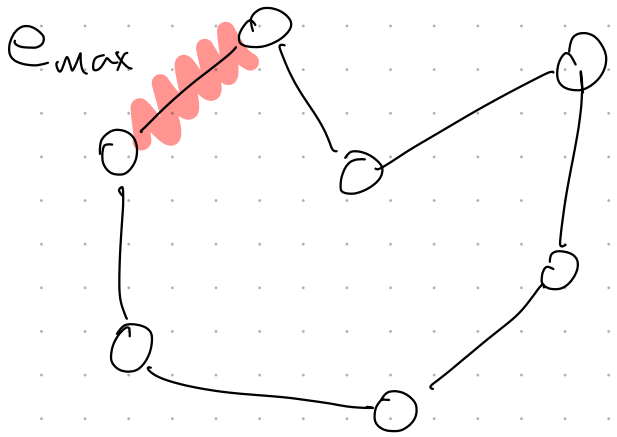
The Cycle Lemma.

$G = (V, E, W)$ w/ distinct edge weights.

Suppose C is a cycle within G and let e_{max} be the edge in C of maximum weight.

Then,

e_{max} is NOT in the MST of G.



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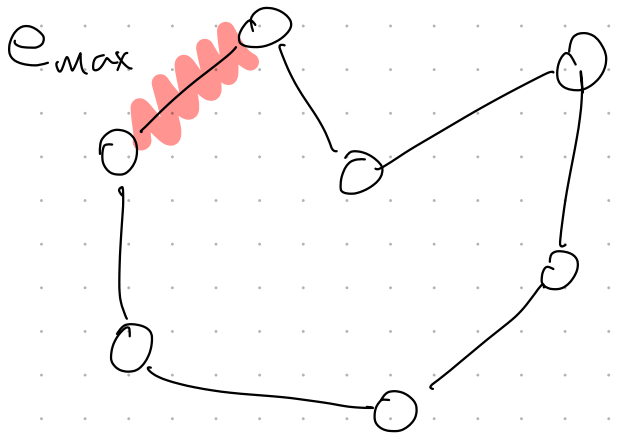

Corollary. DeleteMax returns the MST of G.

↳ KT "Reverse Delete"

Proof of Correctness of Delete Max (assuming Cycle Lemma)

Delete Max

$T = (V, E)$

For $i = m \longrightarrow 1$.

  if $T \setminus \{e_i\}$ is still connected

    Remove e_i from T

Return T.

Proof of Correctness of Delete Max (assuming Cycle Lemma)

By contradiction,

Suppose the MST is T^*, but Delete Max returns $T \neq T^*$

$\Rightarrow$ Delete Max deletes some edge $(u, v) \in T^*$.

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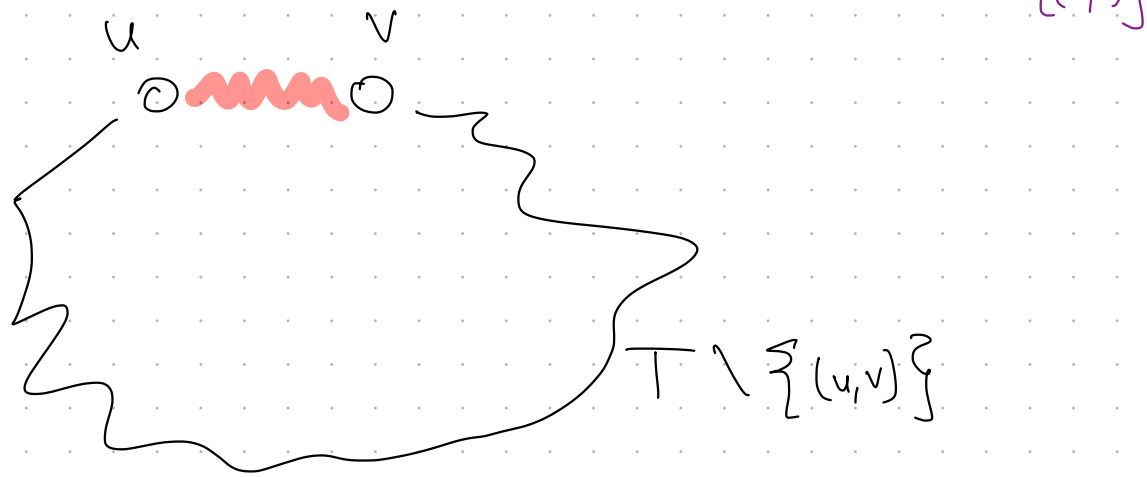
$\Rightarrow$ Delete Max deletes some edge $(u,v) \in T^*$.

When (u,v) is removed from T,

$\exists$ a path p from $u \rightarrow v$ in $T \setminus \{(u,v)\}$



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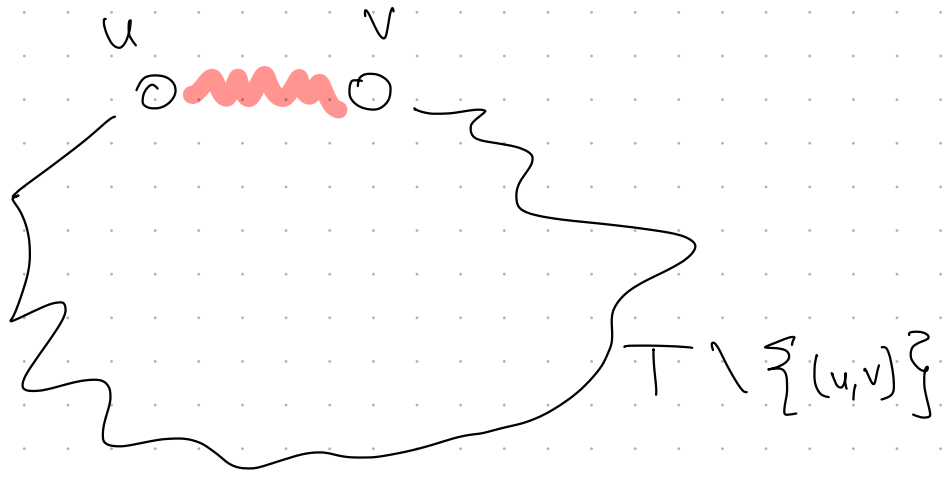

$\Rightarrow p \cup \{(u,v)\}$ is a cycle in G

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By contradiction,

Suppose the MST is T^*, but Delete Max returns $T \neq T^*$

$\Rightarrow$ Delete Max deletes some edge $(u,v) \in T^*$.

$\Rightarrow$ $P \cup \{(u,v)\}$ is a cycle in G

But why did we remove (u,v)?

Delete Max

$T = (V, E)$

For $i = m \longrightarrow 1$.

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Return T.

By removal order (greatest → least)

(u,v) is the max wt. edge in cycle!

Proof of Correctness of Delete Max (assuming Cycle Lemma)

By contradiction,

Suppose the MST is T^*, but Delete Max returns $T \neq T^*$

$\Rightarrow$ Delete Max deletes some edge $(u,v) \in T^*$.

$\Rightarrow$ $P \cup \{(u,v)\}$ is a cycle in G

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$T = (V, E)$

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Return T.

By removal order (greatest → least)

(u,v) is the max wt. edge in cycle!

$\Rightarrow$ By the Cycle Lemma

$(u,v) \notin T^*$

(A contradiction)

So, to complete proof of correctness for DeleteMax we need to prove the Cycle Lemma.

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Assumptions

1. $G = (V, E, W)$ w/ distinct edge wts.
2. C is a cycle in G containg e_{max}
3. e_{max} is the max wt edge on C

Conclusion

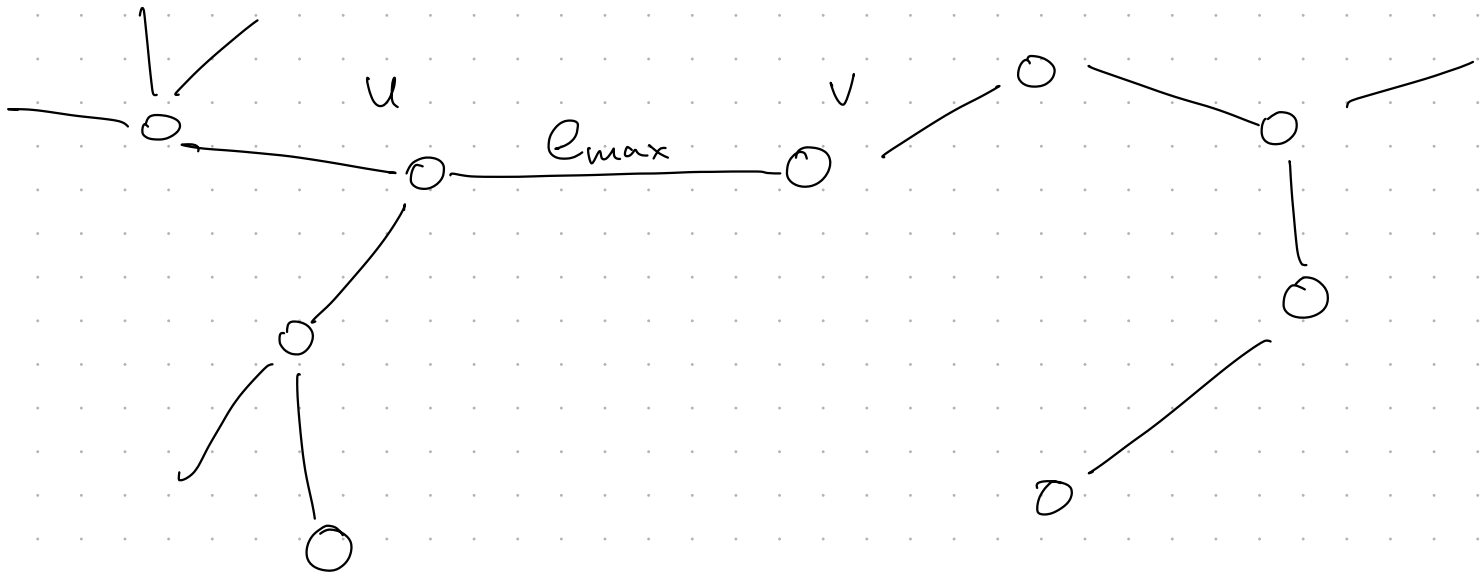
* e_{max} is NOT in MST.

Next. Exchange Argument

Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

We show that T is NOT the minimum spanning tree.

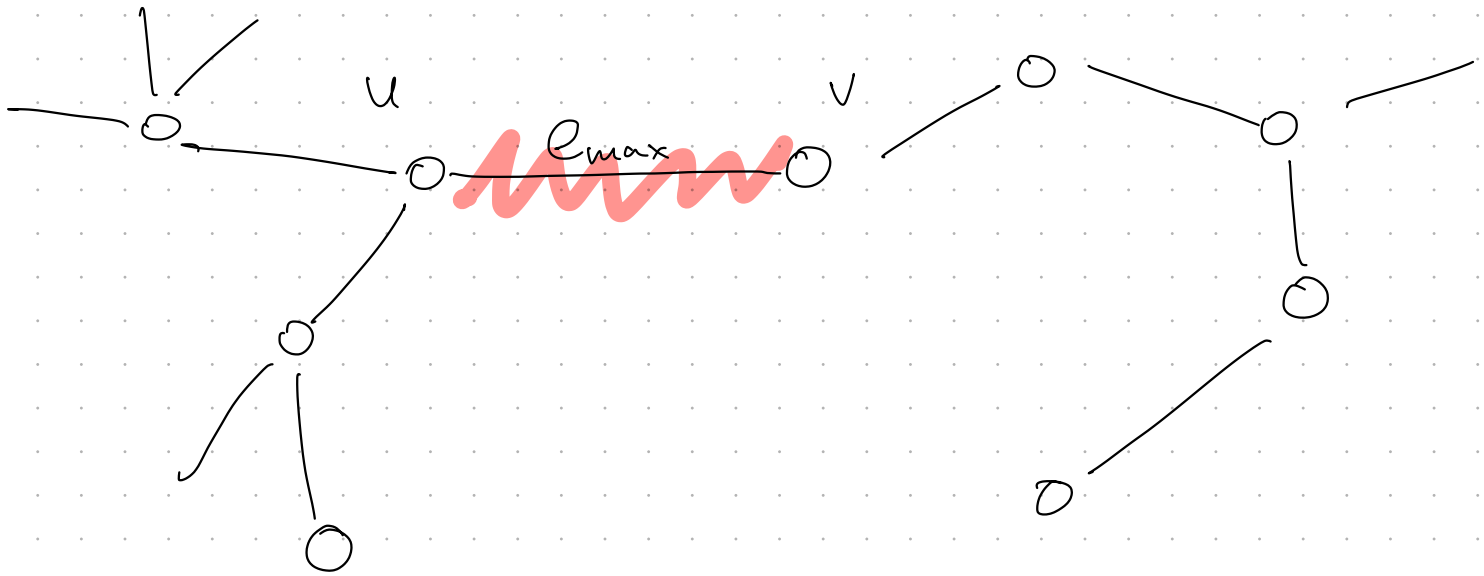


Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

We show that T is NOT the minimum spanning tree.

Consider removing e_{max} from T.



Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

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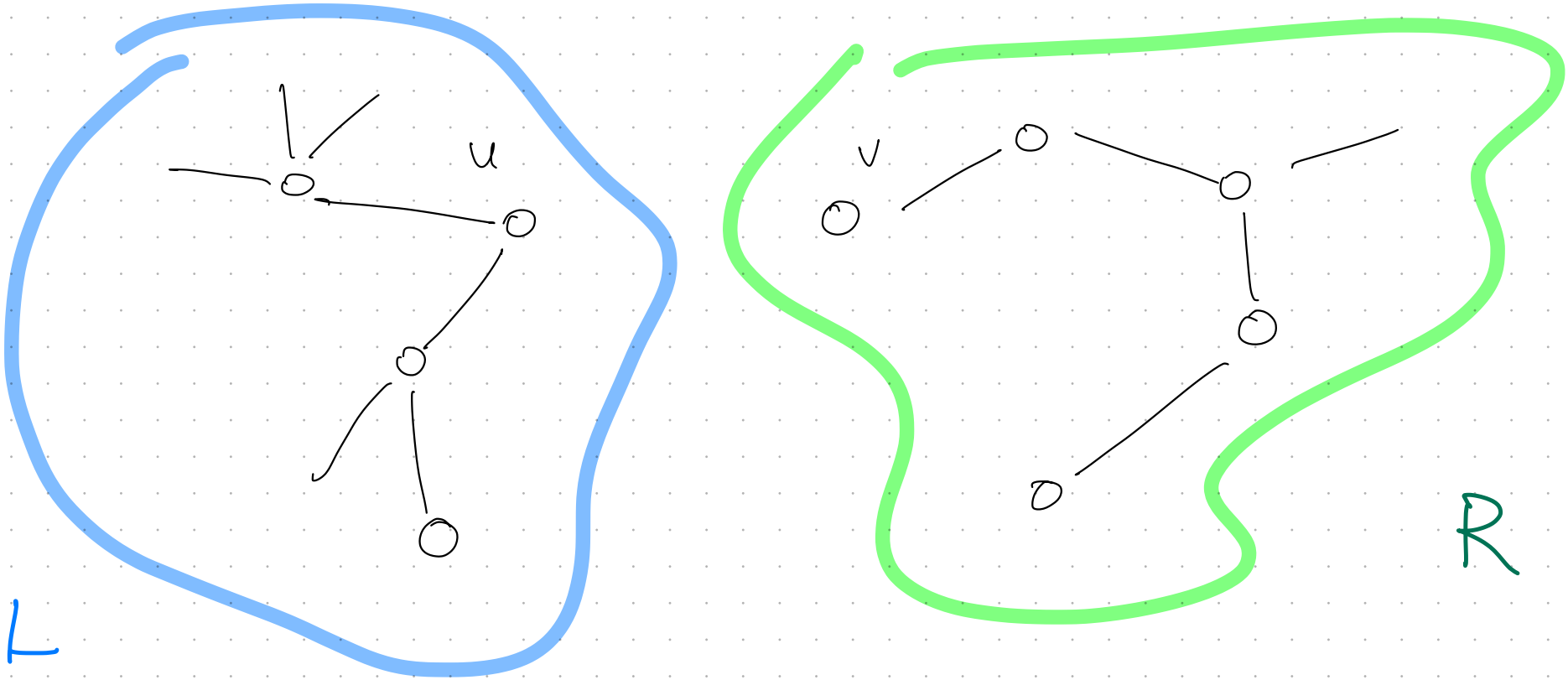
Disconnects T into two pieces.



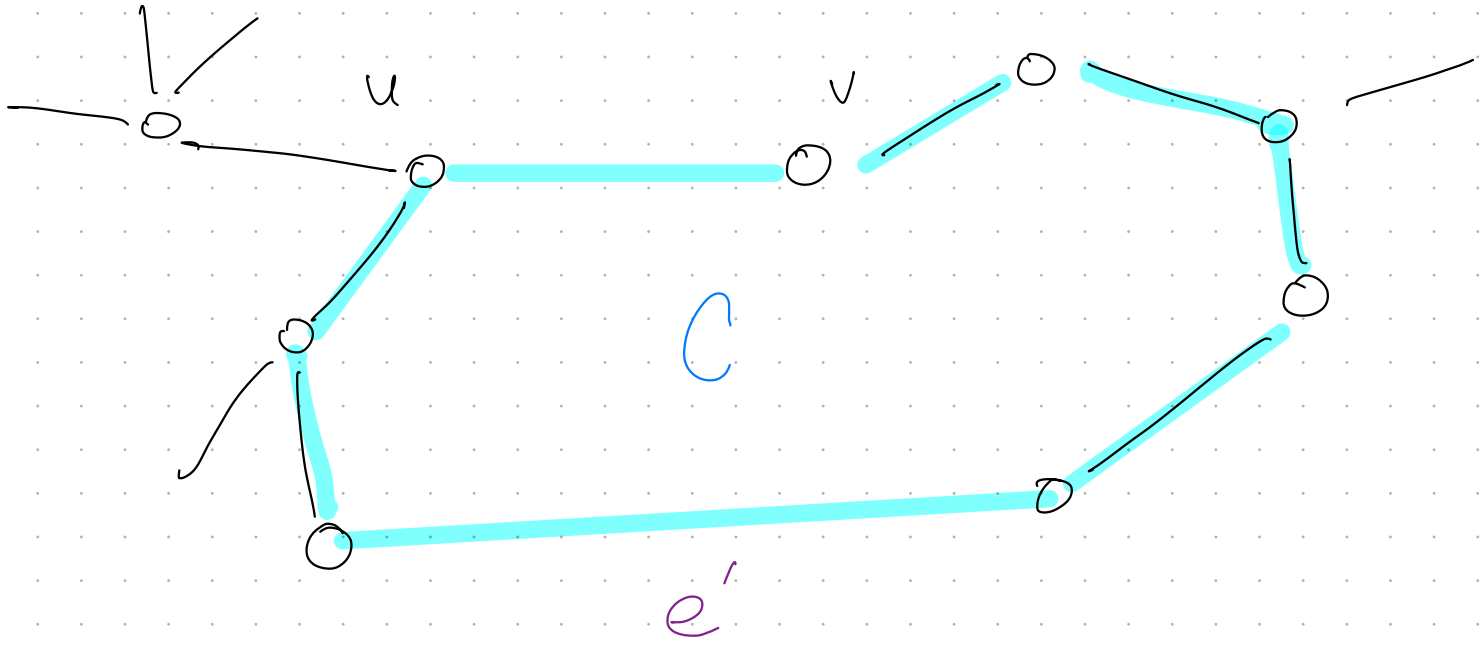
Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

We show that T is NOT the minimum spanning tree.

By (2), e_{max} is in some cycle C within G.

$\Rightarrow$ there exists some edge $e' \in C$ (but not in T) from L to R



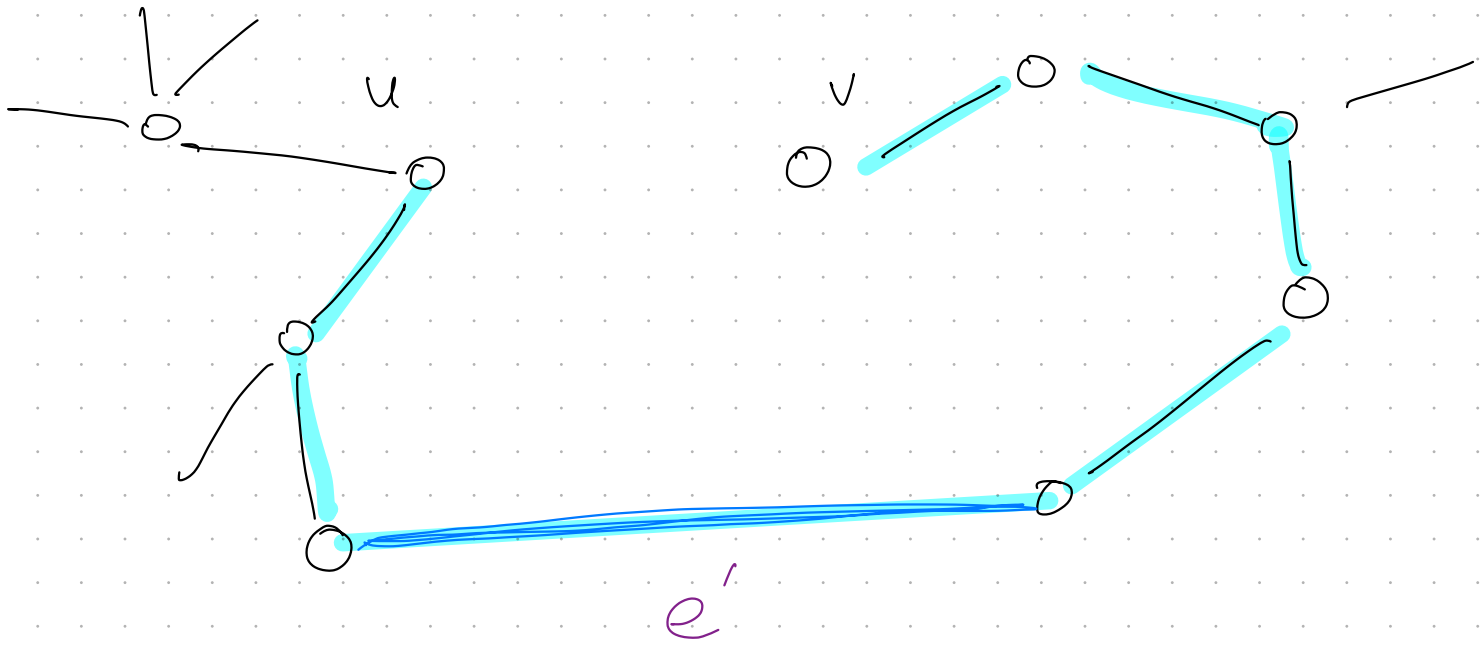
Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

We show that T is NOT the minimum spanning tree.

Consider exchanging e' for e_{max}.

$$T' = T \setminus \{e_{max}\} \cup \{e'\}$$



Claims.

① T' has smaller weight than T

② T' is a tree

③ T' is connected (i.e. a spanning tree)

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① T' has smaller weight than T

- e_{max} is the max wt. edge on C
- exchanged for $e' \in C$.

$\Rightarrow w_{T'} < w_T$ by ① & ③

② T' is a tree

③ T' is connected (i.e. a spanning tree)

Claims.

① T' has smaller weight than T

② T' is a tree

- if we add e' to T, we only introduce 1 cycle
- Removing e_{max} breaks this cycle.

③ T' is connected (i.e. a spanning tree)

Claims.

① T' has smaller weight than T

② T' is a tree

③ T' is connected (i.e. a spanning tree)

- Original T was spanning.
- Any path using e_{max} can use detour along C to go from $u \leftrightarrow v$.

Pf. By exchange argument.

Suppose T is a spanning tree s.t. $e_{max} = (u, v) \in T$

We show that T is NOT the minimum spanning tree.

Thus, T' is a spanning tree w/ $w_{T'} < w_T$

So T is NOT the MST. ▨