CS412/413

Introduction to Compilers
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Lecture 21: More About Dataflow Analysis
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Lattices

- Lattice:
  - Set augmented with a partial order relation $\sqsubseteq$
  - Each subset has a LUB and a GLB
  - Can define: meet $\sqcap$, join $\sqcup$, top $\top$, bottom $\perp$
- Use lattice in the compiler to express information about the program
- To compute information: build constraints which describe how the lattice information changes
  - Effect of instructions: transfer functions
  - Effect of control flow: meet operation

Transfer Functions

- Let L = dataflow information lattice
- Transfer function $F_I : L \rightarrow L$ for each instruction I
  - Describes how I modifies the information in the lattice
  - If in[I] is info before I and out[I] is info after I, then
    Forward analysis: out[I] = F_I(in[I])
    Backward analysis: in[I] = F_I(out[I])
- Transfer function $F_B : L \rightarrow L$ for each basic block B
  - Is composition of transfer functions of instructions in B
  - If in[B] is info before B and out[B] is info after B, then
    Forward analysis: out[B] = F_B(in[B])
    Backward analysis: in[B] = F_B(out[B])

Monotonicity and Distributivity

- Two important properties of transfer functions
- Monotonicity: function $F : L \rightarrow L$ is monotonic if
  $x \sqsubseteq y$ implies $F(x) \sqsubseteq F(y)$
- Distributivity: function $F : L \rightarrow L$ is distributive if
  $F(x \sqcap y) = F(x) \sqcap F(y)$
- Property: F is monotonic iff $F(x \sqcap y) \sqsubseteq F(x) \sqcap F(y)$
  - any distributive function is monotonic!

Proof of Property

- Prove that the following are equivalent:
  1. $x \sqsubseteq y$ implies $F(x) \sqsubseteq F(y)$, for all x, y
  2. $F(x \sqcap y) \sqsubseteq F(x) \sqcap F(y)$, for all x, y
- Proof for "1 implies 2"
  - Need to prove that $F(x \sqcap y) \sqsubseteq F(x)$ and $F(x \sqcap y) \sqsubseteq F(y)$
  - Use $x \sqcap y \sqsubseteq x$, $x \sqcap y \sqsubseteq y$, and property 1
- Proof of "2 implies 1"
  - Let x, y such that $x \sqsubseteq y$
  - Then $x \sqcap y = x$, so $F(x \sqcap y) = F(x)$
  - Use property 2 to get $F(x) \sqsubseteq F(x) \sqcap F(y)$
  - Hence $F(x) \sqsubseteq F(y)$

Control Flow

- Meet operation models how to combine information at split/join points in the control flow
  - If in[B] is info before B and out[B] is info after B, then:
    Forward analysis: in[B] = $\sqcap$ {out[B'] | B'$\in$pred(B)}
    Backward analysis: out[B] = $\sqcap$ {in[B'] | B'$\in$succ(B)}
- Can alternatively use join operation $\sqcup$ (equivalent to using the meet operation $\sqcap$ in the reversed lattice)

Monotonicity of Meet

- Meet operation is also monotonic over L x L:

  x1 ⊑ y1 and x2 ⊑ y2 implies (x1 ⊓ x2) ⊑ (y1 ⊓ y2)

- Proof:
  - any lower bound of {x1,x2} is also a lower bound of {y1,y2}, because x1 ⊑ y1 and x2 ⊑ y2
  - x1 ⊓ x2 is a lower bound of {x1,x2}
  - So x1 ⊓ x2 is a lower bound of {y1,y2}
  - But y1 ⊓ y2 is the greatest lower bound of {y1,y2}
  - Hence (x1 ⊓ x2) ⊑ (y1 ⊓ y2)

Forward Dataflow Analysis

- Control flow graph G with entry (start) node B_s
- Lattice (L, ⊑) represents information about program
  - Meet operator ⊓, top element ⊤
- Monotonic transfer functions
  - Transfer function F_I:L → L for each instruction I
  - Can derive transfer functions F_B for basic blocks
- Goal: compute the information at each program point, given the information at entry of B_s is X_0
- Require the solution to satisfy:

  out[B] = F_B(in[B]), for all B

  in[B] = ⊓ {out[B′] | B′∈pred(B)}, for all B

  in[B_s] = X_0

Backward Dataflow Analysis

- Control flow graph G with exit node B_e
- Lattice (L, ⊑) represents information about program
  - Meet operator ⊓, top element ⊤
- Monotonic transfer functions
  - Transfer function F_I:L → L for each instruction I
  - Can derive transfer functions F_B for basic blocks
- Goal: compute the information at each program point, given the information at exit of B_e is X_0
- Require the solution to satisfy:

  in[B] = F_B(out[B]), for all B

  out[B] = ⊓ {in[B′] | B′∈succ(B)}, for all B

  out[B_e] = X_0

Dataflow Equations

- The constraints are called dataflow equations:

  out[B] = F_B(in[B]), for all B

  in[B] = ⊓ {out[B′] | B′∈pred(B)}, for all B

  in[B_s] = X_0

- Solve equations: use an iterative algorithm
  - Initialize in[B_S] = X_0
  - Initialize everything else to ⊤
  - Repeatedly apply rules
  - Stop when reach a fixed point

Algorithm

```
in[B_S] = X_0
out[B] = ⊤, for all B

Repeat
    For each basic block B ≠ B_s
        in[B] = ⊓ {out[B′] | B′∈pred(B)}
    For each basic block B
        out[B] = F_B(in[B])
Until no change
```


Efficiency

- Algorithm is inefficient
  - Effects of basic blocks re-evaluated even if the input information has not changed
- Better: re-evaluate blocks only when necessary
- Use a worklist algorithm
  - Keep of list of blocks to evaluate
  - Initialize list to the set of all basic blocks
  - If out[B] changes after evaluating out[B] = F_B(in[B]), then add all successors of B to the list

Worklist Algorithm

$in[B_S] = X_0$
$out[B] = \top$, for all B
worklist = set of all basic blocks B

Repeat
  Remove a node B from the worklist
  $in[B] = \sqcap \{out[B'] \mid B' \in pred(B)\}$
  $out[B] = F_B(in[B])$
  if out[B] has changed, then
    worklist = worklist $\cup$ succ(B)
Until worklist = $\emptyset$

Correctness

- Initial algorithm is correct
  - If dataflow information does not change in the last iteration, then it satisfies the equations
- Worklist algorithm is correct
  - Maintains the invariant that
    $in[B] = \sqcap \{out[B'] \mid B' \in pred(B)\}$
    $out[B] = F_B(in[B])$
    for all the blocks B not in the worklist
  - At the end, worklist is empty

Termination

- Do these algorithms terminate?
- Key observation: at each iteration, information decreases in the lattice
  $in_{k+1}[B] \sqsubseteq in_k[B]$ and $out_{k+1}[B] \sqsubseteq out_k[B]$
  where $in_k[B]$ is info before B at iteration k and $out_k[B]$ is info after B at iteration k
- Proof by induction:
  - Induction basis: true, because we start with top element, which is greater than everything
  - Induction step: use monotonicity of transfer functions and meet operation
- Information forms a chain: $in_1[B] \sqsupseteq in_2[B] \sqsupseteq in_3[B] \ldots$

Chains in Lattices

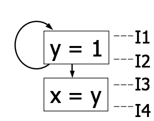
- A chain in a lattice L is a totally ordered subset S of L:
  $x \sqsubseteq y$ or $y \sqsubseteq x$ for any $x, y \in S$
- In other words:
  Elements in a totally ordered subset S can be indexed to form an ascending sequence:
  $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \ldots$
  or they can be indexed to form a descending sequence:
  $x_1 \sqsupseteq x_2 \sqsupseteq x_3 \sqsupseteq \ldots$
- Height of a lattice = size of its largest chain
- Lattice with finite height: only has finite chains

Termination

- In the iterative algorithm, for each block B:
  $\{in_1[B], in_2[B], \ldots\}$
  is a chain in the lattice, because transfer functions and meet operation are monotonic
- If lattice has finite height then these sets are finite, i.e. there is a number k such that $in_i[B] = in_{i+1}[B]$, for all $i \geq k$ and all B
- If $in_i[B] = in_{i+1}[B]$ then also $out_i[B] = out_{i+1}[B]$
- Hence algorithm terminates in at most k iterations
- To summarize: dataflow analysis terminates if
  1. Transfer functions are monotonic
  2. Lattice has finite height

Multiple Solutions

- The iterative algorithm computes a solution of the system of dataflow equations
- … is the solution unique?
- No, dataflow equations may have multiple solutions !
- Example: live variables



Equations: I1 = I2-{y}
I3 = (I4-{x}) U {y}
I2 = I1 U I3
I4 = {}

Solution 1: I1={}, I2={y}, I3={y}, I4={}
Solution 2: I1={x}, I2={x,y}, I3={y}, I4={}

Safety

- Solution for live variable analysis:
  - Sets of live variables must include each variable whose values will further be used in some execution
  - … may also include variables never used in any execution!
- The analysis is safe if it takes into account all possible executions of the program
  - … may also characterize cases which never occur in any execution of the program
  - Say that the analysis is a conservative approximation of all executions
- In example
  - Solution 2 includes x in live set I1, which is not used later
  - However, analysis is conservative

Safety and Precision

- Safety: dataflow equations guarantee a safe solution to the analysis problem
- Precision: a solution to an analysis problem is more precise if it is less conservative
- Live variables analysis problem:
  - Solution is more precise if the sets of live variables are smaller
  - Solution which reports that all variables are live at each point is safe, but is the least precise solution
- In the lattice framework: S1 is less precise than S2 if the result in S1 at each program point is less than the corresponding result in S2 at the same point
  - Use notation $S1 \sqsubseteq S2$ if solution S1 is less precise than S2

Maximal Fixed Point Solution

- Property: among all the solutions to the system of dataflow equations, the iterative solution is the most precise
- Intuition:
  - We start with the top element at each program point (i.e. most precise information)
  - Then refine the information at each iteration to satisfy the dataflow equations
  - Final result will be the closest to the top
- Iterative solution for dataflow equations is called Maximal Fixed Point solution (MFP)
- For any solution FP of the dataflow equations: $FP \sqsubseteq MFP$

Meet Over Paths Solution

- Is MFP the best solution to the analysis problem?
- Another approach: consider a lattice framework, but use a different way to compute the solution
  - Let G be the control flow graph with start block B_0
  - For each path $p_n=[B_0, B_1, \ldots, B_n]$ from entry to block B_n:
    $in[p_n] = F_{Bn-1} (\ldots (F_{B1}(F_{B0}(X0))))$
  - Compute solution as
    $in[B_n] = \sqcap \{ in[p_n] \mid \text{all paths } p_n \text{ from } B_0 \text{ to } B_n\}$
- This solution is the Meet Over Paths solution (MOP)

MFP versus MOP

- Precision: can prove that MOP solution is always more precise than MFP
  $MFP \sqsubseteq MOP$
- Why not use MOP?
- MOP is intractable in practice
  1. Exponential number of paths: for a program consisting of a sequence of N if statement, there will 2^N paths in the control flow graph
  2. Infinite number of paths: for loops in the CFG

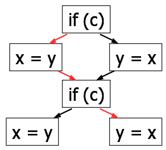
Importance of Distributivity

- Property: if transfer functions are distributive, then the solution to the dataflow equations is identical to the meet-over-paths solution
  $MFP = MOP$
- For distributive transfer functions, can compute the intractable MOP solution using the iterative fixed-point algorithm

Better Than MOP?

- Is MOP the best solution to the analysis problem?

- MOP computes solution for all path in the CFG
- There may be paths which will never occur in any execution
- So MOP is conservative



- IDEAL = solution which takes into account only paths which occur in some execution

- This is the best solution
- … but it is undecidable

Summary

- Dataflow analysis
  - sets up system of equations
  - iteratively computes MFP
  - Terminates because transfer functions are monotonic and lattice has finite height

- Other possible solutions: FP, MOP, IDEAL
- All are safe solutions, but some are more precise:

  FP $\sqsubseteq$ MFP $\sqsubseteq$ MOP $\sqsubseteq$ IDEAL
- MFP = MOP if distributive transfer functions

- MOP and IDEAL are intractable
- Compilers use dataflow analysis and MFP