CS 312
Type Inference

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Type Checking vs Inference

- Type checking: given an expression e, and a type environment E for the free variables in e, check if e is well formed and return its type

```
(* Returns the type of exp in environment E
 * Raises: TypeError if exp is not well-typed *)
fun type_check(E:environment, exp:expression): typ
```

- Type inference: given an expression e, compute the types of all variables in e. Use those types to determine the type of e and its sub-expression
  - Goal: identify the most general (i.e. polymorphic) type of e

Type Inference

- General approach: constraint-based formulation
- Step 1: Assign fresh types (i.e., type variables)
- Step 2: Generate constraints between type variables by recursively walking the expression tree
  - Generate more fresh type variables
- Step 3: Solve constraints between type variables to infer the types

Example

```
fun map (f, l) =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
end
```

(An example taken from David Walker @ Princeton)

Step 1: Fresh Types for Variables

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
end
```


Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
end
```

null: b' list -> bool

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l : b′ list) : bool then
    nil
  else
    cons (f (hd l), map (f, tl l)))
end

null: b′ list -> bool

constraints
b = b′ list

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l) then
    nil : d list
  else
    cons (f (hd l), map (f, tl l)))
end

constraints
b = b′ list

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l) then
    nil : d list
  else
    cons (f (hd l), map (f, tl l)))
end

hd: b″ list -> b″
tl: b‴ list -> b‴ list

constraints
b = b′ list

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l) then
    nil : d list
  else
    cons (f (hd l :b″), map (f, tl l :b‴ list)))
end

b = b″ list
b = b‴ list

constraints
b = b′ list
b = b″ list
b = b‴ list

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l) then
    nil : d list
  else
    cons (f (hd l :b″), map (f :a, tl l :b‴ list)))
end

constraints
b = b′ list
b = b″ list
b = b‴ list

Step 2: Generate Constraints

fun map (f : a, l : b) : c =
  if null (l) then
    nil : d list
  else
    cons (f (hd l :b″), map (f :a, tl l :b‴ list)))
end

f : a
map: a -> b -> c

constraints
b = b′ list
b = b″ list
b = b‴ list

Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   : d
   else
      cons (f (hd l :b''), map (f :a, tl l :b''' list)))
end
```

f : a

map: a -> b -> c

a = b'' -> a'

a=a
b = b''' list

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   : d list
   else
      cons (f (hd l) :a', map (f, tl l) :c))
end
```

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   :
   else
      cons (f (hd l) :a', map (f, tl l) :c))
end
```

cons: c' * c' list -> c' list

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   :
   else
      cons (f (hd l) :a', map (f, tl l) :c)) :c' list
end
```

cons: c' * c' list -> c' list

a' = c'

c = c' list

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   : d list
   else
      cons (f (hd l), map (f, tl l))) :c' list
end
```

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'
a' = c'
c = c' list

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil   : d list
   else
      cons (f (hd l), map (f, tl l))) :c' list
end
```

d list = c' list

constraints
b = b' list
b = b'' list
b = b''' list
a = b'' -> a'
a' = c'
c = c' list
d list = c' list

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Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
   : d list
end
```

constraints
b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list

Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
   : d list
end
```

d list = c

constraints
b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Step 2: Generate Constraints

```
fun map (f : a, l : b) : c =
   if null (l) then
      nil
   else
      cons (f (hd l), map (f, tl l)))
end
```

constraints
b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Step 3: Solve Constraints

- Constraint solution provides all possible solutions to type annotations on terms

constraints
b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

⇨

solution
a = b′ -> c′
b = b′ list
c = c′ list

⇨

```
map (f : 'b->'c
     x : 'b list)
: 'c list
=
 ...
end
```


Solving Constraints

- Iterative constraint solving using substitutions (Martelli-Montanari algorithm): compute a map from type variables to most general types
  - Iterate through the constraints
  - Replace t1->t2 = t3->t4 by t1 = t3, t2 = t4
  - Replace list t1 = list t2 by t1 = t2
  - If t1 = t2 where t1 and t2 are incompatible types (e.g. t1 is is a function type and t2 is list type) then Type Error
  - Constraint a = t, where a is a type variable
    - If a occurs in t then "Type Error"
    - If t is a then discard the constraint
    - Otherwise, add a → t to the mapping and substitute t for a in all other constraints

Example

Constraints	Mappings
b = b′ list	
b = b″ list	
b = b‴ list	
a = b″ -> a′	
a′ = c′	
c = c′ list	
d list = c′ list	
d list = c	

Example

Constraints

b = b′ list
b′ list = b″ list
b′ list = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Mappings

b → b′ list

Example

Constraints

b′ list = b″ list
b′ list = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Mappings

b → b′ list

Example

Constraints

b′ = b″
b″ list = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Mappings

b → b′ list
b′ → b″

(the process continues…)

Example

Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Solution

b → b′ list
b′ → b″
b″ → b‴
a → b‴ -> a′
a′ → c′
c → c′ list
d → c′

Example

- The mapping essentially forms a DAG (directed acyclic graph)
- To find the type for a variable, traverse the DAG
- The types a, b, c can be expressed in terms of b’’’ and c’:

a = b‴ -> c′
b = b‴ list
c = c′ list

Solution

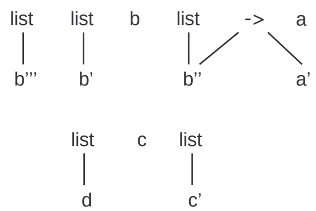
b → b′ list
b′ → b″
b″ → b‴
a → b‴ -> a′
a′ → c′
c → c′ list
d → c′

Unifications

- Another way of solving constraints: via unifications
- Consider a graph where each node is either a type variable, or a function type, or a list type.
- Function types have two child types; list types have one child.

- Solve the constraint t = t’ by unifying the types of t and t’ in the graph
- Recursive unifications: unify(t1->t2, t3->t4) implies unify(t1,t3) and unify(t3,t4)

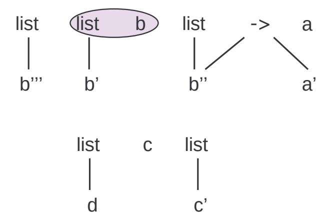
Example



Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

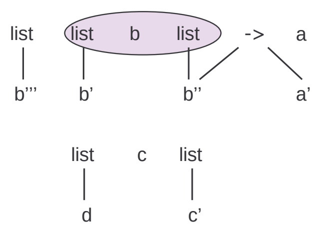
Example



Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

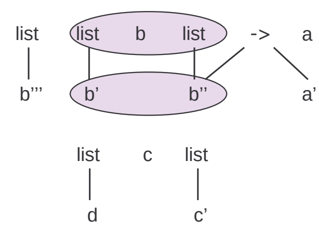
Example



Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

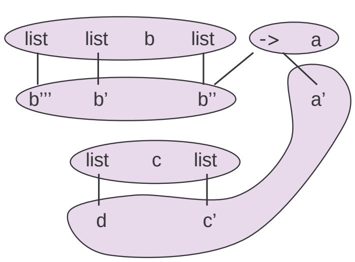
Example



Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

Solution



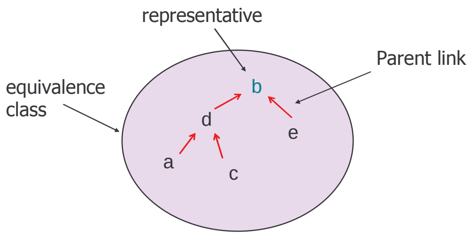
Constraints

b = b′ list
b = b″ list
b = b‴ list
a = b″ -> a′
a′ = c′
c = c′ list
d list = c′ list
d list = c

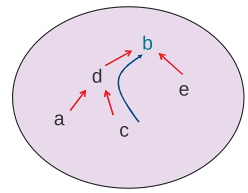
Union-Find Data Structure

- Unifications can be implemented efficiently using union-find data structures.
- Models equivalence classes:
  - Each class has a representative
  - Each node has a parent; the node without a parent is the representative
- Supports two operations:
  - Union: merges together two equivalence classes
  - Find: lookup the representative of a class.

Union-Find Data Structure



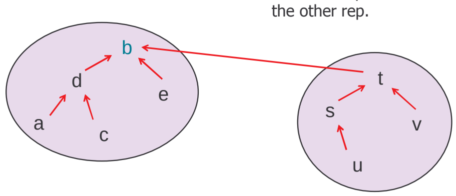
Find Operation



Find operation: traverse parent pointers up

Union Operation: O(1)

Union operation:
Choose one of the reps
Make it the parent of
the other rep.



Union-Find Data Structure