- Diagnosis
  (specific: do I need example)
  - when to use diag?

- Negating statements
  - are there multiple answers?

✓ Proofs that f^-1 (inverses) exist
  - what can you define/use.
  - relationships between f, f^-1, injectivity

- Example: are all finite sets countable?

✓ Cardinality
  - def's
  - \( \subseteq \) vs. \( \leq \) vs. \( \neq \) vs. \( \neq \)
  - proving \( X \) countable.
  - Proving uncountable by showing \( |X| > |\mathbb{N}| \)?

- Equivalence relations
  - def's, examples.
Inverses & Injectivity

\[ f \text{ has a left inverse iff } f \text{ is injective.} \]
\[ f \text{ has a right inverse iff } f \text{ is surjective.} \]
\[ f \text{ has a 2-sided inverse iff } f \text{ is bijective.} \]

Claim: If \( f \) is injective then \( f \) has a left inverse.

Proof: Assume \( f: A \rightarrow B \) is injective.

We say \( f \) has a left inverse.

Let \( g: B \rightarrow A \) be given as follows:

- If \( y = f(x) \) for some \( x \), let \( g(y) \) be chosen any \( x \in A \).
- If \( y \) is not an output of \( f \), let \( g(y) \) be \( x_0 \) with \( x_0 \neq x \).

Does \( g \) give an output for every input?

Yes.

If \( y = f(x_1) \) and \( y = f(x_2) \), then \( f(x_1) = f(x_2) \) so (since \( f \) is injective) \( x_1 = x_2 \).
Diagonalization

When to diagonalize?
- When you want to show $X$ is uncountable.

Ex: $\mathbb{R}$ is the set of real #s between 0 & 1 is uncountable.

Pf: Assume $X$ is countable. Then $|\mathbb{N}| \geq |X|$, so $\exists f: \mathbb{N} \to X$ that is surjective.

$f$ might be drawn:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000...</td>
</tr>
<tr>
<td>1</td>
<td>0.5000...</td>
</tr>
<tr>
<td>2</td>
<td>0.1100...</td>
</tr>
<tr>
<td>3</td>
<td>0.2111...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$x_0$ = 0.5566...

$x_0$ can't be equal to $f(n)$ because the $n$th digit after the decimal point is different.

This says $f$ never outputs $x_0$, so $f$ can't be surjective, so this is a contradiction.

F is surjective means $\forall x \in X$, $\exists n \in \mathbb{N}$, $f(n) = x$.
F is not surjective means $\exists x \in X$, $\forall n \in \mathbb{N}$, $f(n) \neq x$. 

Defn: $X$ is countable means $|\mathbb{N}| \geq |X|$. 
$X$ is countable means $|N| \geq |X|$ means $\exists f: N \to X$ surjective

$\mathbb{Z}$ is countable
$f: N \to \mathbb{Z}$

$f$ is clearly surjective, because every $n \in \mathbb{Z}$ is output by $f$

$N \times N$ is countable

$f(k) = n$ if pair found when traversing diagonal as in picture.

$|A| \geq |B|$ or $|A| > |B|

\{ |A| \geq |B| \text{ and } |A| \neq |B| \}

\text{good descriptors:}