

Combinatorial proofs

Claim: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Let A be a set with n elements.

Let $X = \{B \subseteq A \mid |B| = k\}$. We've seen that $|X| = \binom{n}{k}$.

Let $Y = \{C \subseteq A \mid |C| = n - k\}$. We've seen that $|Y| = \binom{n}{n-k}$.

Let $f : X \rightarrow Y$ be given by $f(B) = A \setminus B$. Clearly f is a bijection. Therefore $LHS = |X| = |Y| = RHS$, as required.

Claim: $2^n = \sum_{k=0}^n \binom{n}{k}$

Proof: Let A be a set with n elements.

Let $X = 2^A$. We've seen that $|X| = 2^n$

Let $Y = 2^A$. We can construct an element of Y by first choosing $k \in [0..n]$, and then choosing a subset $B \subseteq A$ of size k (there are $\binom{n}{k}$ options). By the sum rule, we have $|Y| = \sum_{k=0}^n \binom{n}{k}$.

But $X = Y$ so $LHS = |X| = |Y| = RHS$ as required.

Answers to problems we've solved:

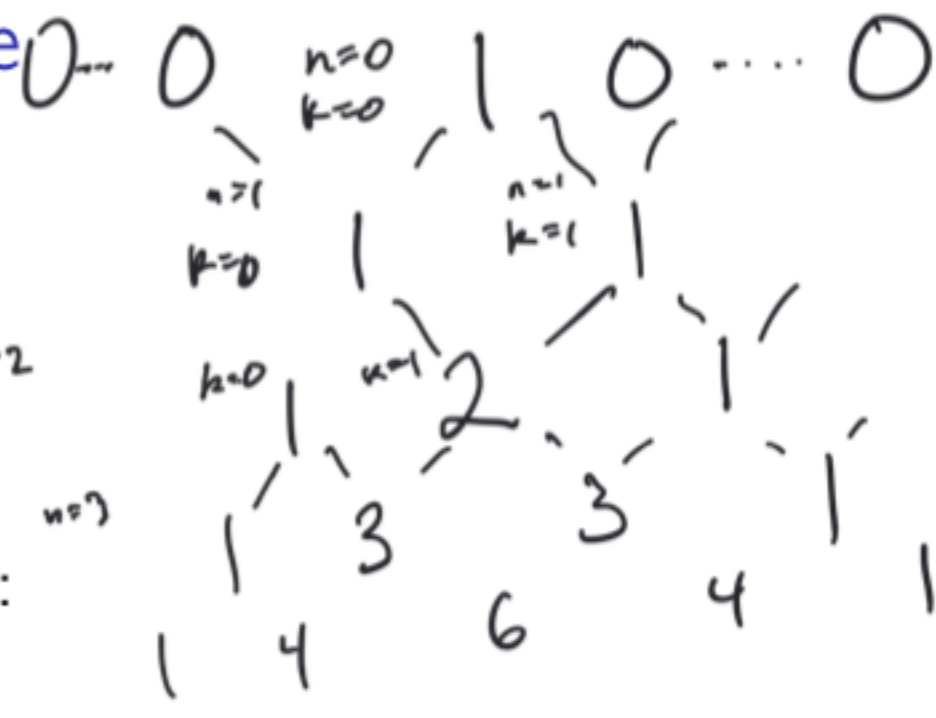
$$A = \{a_1, a_2, \dots, a_a\} \quad |A| = a$$

$$B = \{b_1, \dots, b_b\} \quad |B| = b$$

Number of ways	Process
a	Choose an element of A
ab	Choose an element of A and then an element of B
$a + b$	Choose an element of A or an element of B
$\sum_i a_i$	Choose i and then an element of A_i
2^a	Choose a subset of A
$a!$	Choose a permutation of A
$a!/(a-k)!$	Choose a sequence of k different elements of A
$\binom{a}{k}$	Choose a subset of k elements of A
a/k	Choose an element of A/R where $\forall a, [a] = k$

k-permutations of A.

Another combinatorial proof / pascal's triangle



Claim: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Proof: Let $A = \{a_1, a_2, \dots, a_n\}$

Let $X =$ ^{set of subsets B} of A of size k. Then $|X| = LHS$ because:

(we've seen this before)

Let $Y =$ ^{set of subsets} $B \subset A$ of size k. Then $|Y| = RHS$ because:

I can ^ε form B by either choosing k elts or choosing k-1 elts and a_n .

Now $|X| = |Y|$ because:

$\leftarrow \binom{n-1}{k}$ opts \leftarrow choose k elements from $\{a_1, a_2, \dots, a_{n-1}\} \rightarrow$ output B

$\leftarrow \binom{n-1}{k-1}$ opts \leftarrow choose k-1 elements from $\{a_1, a_2, \dots, a_{n-1}\} \rightarrow$ output $C \cup \{a_n\}$

$X = Y$.

So $LHS = |X| = |Y| = RHS \checkmark$

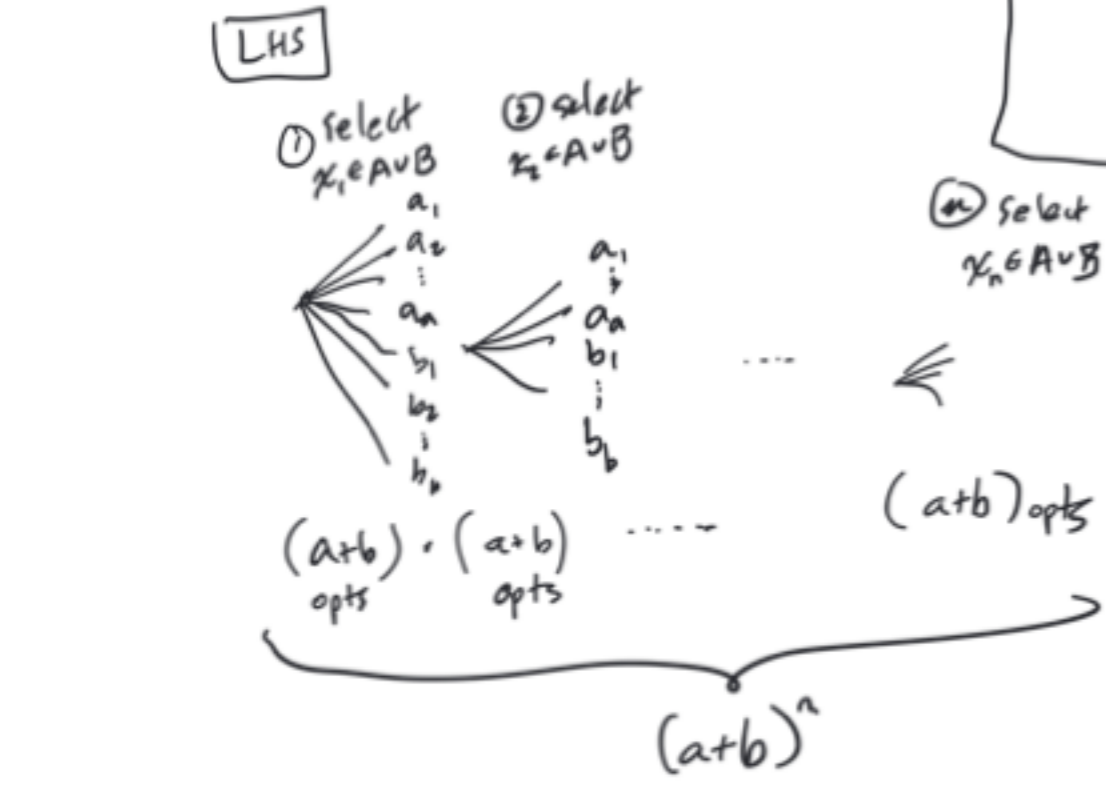
Another combinatorial proof / binomial theorem

$\forall a, b, n$
 Claim: $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

Proof: Let $A = \{a_1, \dots, a_a\}$ $|A|=a$
 Let $B = \{b_1, \dots, b_b\}$ $|B|=b$

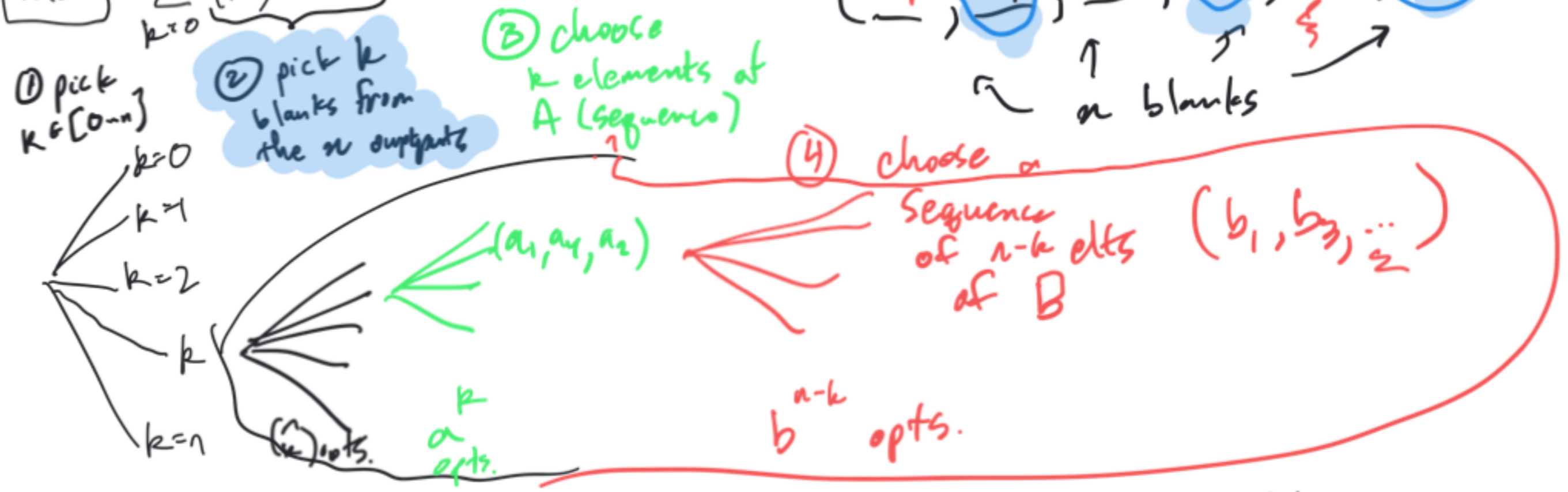
$$(a+b)^4 = \binom{4}{0} a^0 b^4 + \binom{4}{1} a^1 b^3 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a^3 b + \binom{4}{4} a^4 b^0$$

$$= 1 a^0 b^4 + 4 a^1 b^3 + 6 a^2 b^2 + 4 a^3 b + 1 a^4 b^0$$



$|X| = (a+b)^n$

RHS $\sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$



Generates all elements of X exactly once,

So $|X| = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \text{RHS}$

LHS.