Combinatorial proofs

Claim: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Let A be a set with n elements.

Let $X = \{B \subseteq A \mid |B| = k\}$. We've seen that $|X| = \binom{n}{k}$.

Let $Y = \{C \subseteq A \mid |C| = n - k\}$. We've seen that $|Y| = \binom{n}{n-k}$.

Let $f : X \to Y$ be given by $f(B) = A \setminus B$. Clearly f is a bijection. Therefore $LHS = |X| = |Y| = RHS$, as required.

Claim: $2^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k}$

Proof: Let A be a set with n elements.

Let $X = 2^A$. We've seen that $|X| = 2^n$

Let $Y = 2^A$. We can construct an element of Y by first choosing $k \in [0..n]$, and then choosing a subset $B \subseteq A$ of size k (there are $\binom{n}{k}$ options). By the sum rule, we have $|Y| = \sum_{k=0}^{n} \binom{n}{k}$.

But $X = Y$ so $LHS = |X| = |Y| = RHS$ as required.

Answers to problems we've solved:

$A = \{a_1, a_2, \ldots a_a\}$ $|A| = a$
$B = \{b_1, \ldots b_b\}$ $|B| = b$

Number of ways	Process
a	Choose an element of A
ab	Choose an element of A and then an element of B
$a + b$	Choose an element of A or an element of B
$\sum_i a_i$	Choose i and then an element of A_i
2^a	Choose a subset of A
$a!$	Choose a permutation of A
$a!/(a-k)!$	Choose a sequence of k different elements of A
$\binom{a}{k}$	Choose a subset of k elements of A
a/k	Choose an element of A/R where $\forall a, \lvert [a] \rvert = k$

k-permutations of A.

Another combinatorial proof / pascal's triangle

0 … 0 (n=0, k=0) 1 0 … 0

(n=1, k=0) 1 (n=1, k=1) 1

(n=2) (k=0) 1 (k=1) 2 1

(n=3) 1 3 3 1

1 4 6 4 1

Claim: $\underbrace{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$

Proof: Let $A = \{a_1, a_2, \dots, a_n\}$

Let X = set of subsets B of A of size k. Then $|X| = LHS$ because:
(we've seen this before).

Let Y = set of subsets $B \subseteq A$ of size k. Then $|Y| = RHS$ because:

I can form B by either choosing k elts or choosing $k-1$ elts and a_n.

$\binom{n-1}{k}$ opts ← choose k elements from $\{a_1, a_2, \dots, a_{n-1}\}$ → output B

$\binom{n-1}{k-1}$ opts ← choose $k-1$ elements from $\{a_1, a_2, \dots a_{n-1}\}$. → output $C \cup \{a_n\}$.

Now $|X| = |Y|$ because:

$X = Y$.

So $LHS = |X| = |Y| = RHS$ ✓

Another combinatorial proof / binomial theorem

$\forall a,b,n,$

Claim: $\underbrace{(a+b)^n} = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$(a+b)^4 = \binom{4}{0}a^0b^4 + \binom{4}{1}a^1b^3 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^3b + \binom{4}{4}a^4b^0$$

1, 4, 6, 4, 1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Proof: Let $A = \{a_1, \ldots, a_a\}$ $|A| = a$

Let $B = \{b_1, \ldots, b_b\}$ $|B| = b$

$A \cap B = \emptyset$

$\boxed{\text{LHS}}$

① select $x_1 \in A \cup B$: $a_1, a_2, \ldots, a_a, b_1, b_2, \ldots, b_b$

② select $x_2 \in A \cup B$: $a_1, \ldots, a_a, b_1, \ldots, b_b$

……

(n) select $x_n \in A \cup B$

$(a+b)$ opts $\cdot$ $(a+b)$ opts …… $(a+b)$ opts

$\underbrace{\qquad\qquad}_{(a+b)^n}$

output $(x_1, x_2, \ldots, x_n)$

Generates all sequences of n elements where each elt. is in $A \cup B$. X

$|X| = (a+b)^n$

$\boxed{\text{RHS}}$ $\sum_{k=0}^{n} \underbrace{\binom{n}{k} a^k b^{n-k}}$

① pick $k \in [0..n]$: $k=0$, $k=1$, $k=2$, …, k, …, $k=n$

② pick k blanks from the n outputs

$\binom{n}{k}$ opts.

③ choose k elements of A (sequence)

(a_1, a_4, a_2)

a^k opts.

④ choose a sequence of $n-k$ elts of B

$(b_1, b_3, \ldots, \xi)$

b^{n-k} opts.

k slots chosen

$(b_1, a_1, b_3, a_4, \ldots, \xi, \ldots, a_2)$

n blanks

Generates all elements of X exactly once,

So $|X| = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \text{RHS}$

$\shortparallel$

LHS.