Lecture 31: Handling overcounting

Last time: suppose $|A| = k$, $|B| = l$, ..., and all these sets are disjoint. How many ways are there to:

- Select an element of $A$?
- Select an element of $A$ and an element of $B$?
- Select either an element of $A$ or an element of $B$?
- Choose a $0$ and an element of $A_i$, where $|A_i| = n_i$?
- Choose a subset of $A$?
Handling duplicates / ignoring details

**Question:** I have 4 balls: a white, a blue, and a two reds. How many ways are there to order them, if the two red balls are indistinguishable?

1. Choose 1st ball
   - 4 options: 3 reds, 2 reds, 1 red.

2. Choose ball 2
   - 3 choices:
     - w, r₁, r₂ → (w, r₁, r₂, b)
     - r₂, w, r₁ → (r₂, w, r₁, b)
     - r₁, w, r₂ → (r₁, w, r₂, b)

3. Choose ball 3
   - 2 choices:
     - w, b
     - r₂, b

4. Choose ball 4
   - 1 choice: b

Total: \[ \frac{4 \times 3 \times 2 \times 1}{2} = 12 \]
Quotient rule

**Quotient rule:** If \(|A| = n\) and we partition \(A\) into groups, each of size \(k\), then there are \(n/k\) groups.

**Quotient rule in terms of equivalence classes:** If \(|A| = n\) and for each \(a \in A\), \(|[a]_R| = k\) (i.e. \(a\) is equivalent to \(k\) things) then \(|A/R| = n/k\).
Counting sequences

**Question:** If $|A| = n$, how many orderings of $A$ are there?

**Question:** If $|A| = n$, how many orderings of $k$ elements of $A$ are there?

1. \[ \text{# orderings} = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n! \]

2. \[ \text{# orderings} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{(n-k)!} \]

3. \[ \frac{n!}{(n-k)!} \]

**Step 1:** Choose a perm. of $A$

\[ (a_1, a_2, a_3, \ldots) \]

\[ (a_3, a_1, a_2, \ldots) \]

$n!$ opts

- $k = 3$
  - $n = 5$

- $k = 2$
  - $n = 5$

Each output permutation is equivalent to $n! / (n-k)!$ other poss.
Counting subsets

Question: If $|A| = n$, how many subsets of $A$ are there of size $k$?

1. Choose ordering of $k$ elements of $A$.
   \[
   \frac{n!}{(n-k)!}
   \]
   \[(1,2) \not\approx (2,1)\]

2. Each ordering is equivalent to other orderings with same $k$ elts, there $k!$ orderings.

By quotient rule, total # subsets is \[
\frac{n!}{k!(n-k)!} = \binom{n}{k}
\]

"$n$ choose $k$"
Combinatorial proofs

Claim: \( 2^n = \sum_{k=0}^{n} \binom{n}{k} \)

Proof:

Let \( A = \{a_1, a_2, \ldots, a_n\} \). LHS is \( \# \) of subsets of \( A \), because we construct a subset \( B \subseteq A \) by

1. Choose whether \( a_i \in B \), \( a_i \notin B \)
2. Yes \( \Rightarrow \) yes
   \[ a_i \in B \]
   \[ a_i \notin B \]
   \[ \text{no} \]
3. So \( \# \) of subsets of \( A \) is \( 2^n \).
4. Choose \( \binom{n}{k} \) \( \text{sets of size } k \).
5. \( k=0 \Rightarrow \binom{n}{0} \)
6. \( k=1 \Rightarrow \binom{n}{1} \)
7. \( k=n \Rightarrow \binom{n}{n} \)

This process outputs all subsets of \( A \).

So \( \# \) of subsets of \( A \) is \( \sum_{k=0}^{n} \binom{n}{k} \).

So \( 2^n = \sum_{k=0}^{n} \binom{n}{k} \)
Combinatorial proofs

Claim: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: Let \( A \) be a set of size \( n \).

LHS is \( \# \) of \( k \)-element subsets \( B \subseteq A \).

RHS is \( \# \) of \( n-k \)-element subsets \( C \subseteq A \).

Given \( n=6 \):

\[ A = \{1, 2, 3, 4, 5, 6\} \quad k=2 \]

\[ B = \{2, 3, 5\} \]

\[ \text{let } C = A \setminus B \text{; then } C \text{ is a } n-k \text{-element subset of } A \]

\[ C = \{1, 2, 4, 6\} \quad \Rightarrow \quad B = \{3\} \]

So \( f: 2^{\text{subsets}} \rightarrow 2^{\text{subsets}} \) given by \( f(B) = A \setminus B \) is a bijection.

So \( |2^{\text{subsets}}| = |2^{\text{subsets}}| \) so LHS = RHS.