1. True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.

(a) \( \{\emptyset\} = \emptyset \)
(b) Every set is a subset of its power set
(c) A proof that starts “Choose an arbitrary \( y \in \mathbb{N} \), and let \( x = y^2 \)” is likely to be a proof that \( \forall y \in \mathbb{N}, \forall x \in \mathbb{N} \).
(d) The logical negation of “everybody can fool Mike” is “nobody can fool Mike”.
(e) The relation \( \leq \) is an equivalence relation
(f) If there is a bijection from \( \mathbb{N} \times \mathbb{N} \) to \( X \) then \( X \) is countable.
(g) Recall that \( [X \rightarrow Y] \) denotes the set of functions with domain \( X \) and codomain \( Y \). Let \( f : 2^S \rightarrow [S \rightarrow \{0, 1\}] \) be given by \( f(X) := h \) where \( h : S \rightarrow \{0, 1\} \) is given by \( h(s) := 0 \). \( f \) is injective.
(h) \( f \) as just defined is surjective.
(i) If a function has a right inverse, then the right inverse is unique.

2. Briefly and clearly identify the errors in each of the following proofs:

(a) Proof that 1 is the largest natural number: Let \( n \) be the largest natural number. Then \( n^2 \), being a natural number, is less than or equal to \( n \). Therefore \( n^2 - n = n(n-1) \leq 0 \). Hence \( 0 \leq n \leq 1 \). Therefore \( n = 1 \).

(b) Proof that 2 = 1: Let \( a = b \).

\[
\implies a^2 = ab \\
\implies a^2 - b^2 = ab - b^2 \\
\implies (a + b)(a - b) = b(a - b) \\
\implies a + b = b
\]

Setting \( a = b = 1 \), we get \( 2 = 1 \).

(c) Proof that \( (a + b)(a - b) = a^2 - b^2 \): To prove: \( (a + b)(a - b) = a^2 - b^2 \)

\[
\implies a^2 - ab + ab - b^2 = a^2 - b^2 \\
\implies a^2 - b^2 = a^2 - b^2
\]

…which is true, hence the result is proved.

3. Which of these is the correct negation of \( \exists x, \forall y, \exists z, \neg F(x, y, z) \)?

(a) \( \exists x, \exists y, \exists z, F(x, y, z) \)
(b) \( \exists x, \exists y, \exists z, \neg F(x, y, z) \)
(c) \( \forall x, \forall y, \forall z, F(x, y, z) \)
(d) \( \forall x, \forall y, \forall z, \neg F(x, y, z) \)
4. Complete the following diagonalization proof:

**Claim:** $X = [\mathbb{N} \rightarrow \mathbb{N}]$ is uncountable.

**Proof:** We prove this claim by contradiction. Assume that $X$ is countable. Then there exists a function $F : \text{FILL IN}$ that is \text{FILL IN}.

Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of $X$ in a table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0$</td>
<td>$f_0(0)$</td>
<td>$f_0(1)$</td>
<td>$f_0(2)$</td>
<td>...</td>
</tr>
<tr>
<td>$f_1$</td>
<td>$f_1(0)$</td>
<td>$f_1(1)$</td>
<td>$f_1(2)$</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Let $f_D : \text{FILL IN}$ be given by $f_D : x \mapsto \text{FILL IN}$

Then \text{FILL IN}

This is a contradiction because \text{FILL IN}.

5. Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.

(a) The set $\Sigma^*$ containing all finite length strings of 0's and 1's.
(b) The set $2^\mathbb{N}$ containing all sets of natural numbers.
(c) The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.
(d) The set $[\mathbb{N} \rightarrow \{0, 1\}]$ containing all functions from $\mathbb{N}$ to $\{0, 1\}$.

Be sure to include enough detail:

- If listing elements, be sure to clearly state how you are listing them;
- If diagonalizing, be sure it is clear what your diagonal construction is;
- If providing a function, make sure it is clear what the output is on a given input.

6. For any function $f : A \rightarrow B$ and a set $C \subseteq A$, define $f(C) = \{f(x) \mid x \in C\}$. That is, $f(C)$ is the set of images of elements of $C$. Prove that if $f$ is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.

*(Hint: one way to prove this is from the definition of set equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$).*

7. (a) Write the definition of “$f : A \rightarrow B$ is injective” using formal notation ($\forall, \exists$, “and”, “or”, “if . . . then . . .”, $=, \neq, \ldots$).

(b) Similarly, write down the definition of “$f : A \rightarrow B$ is surjective”.

(c) Write down the definition of “$A$ is countable”. You may write “$f$ is surjective” or “$f$ is injective” in your expression.

8. Recall that the composition of two functions $f : B \rightarrow C$ and $g : A \rightarrow B$ is the function $f \circ g : A \rightarrow C$ defined as $(f \circ g)(x) = f(g(x))$. Prove that if $f$ and $g$ are both injective, then $f \circ g$ is injective.

9. For each of the following functions, indicate whether the function $f$ is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.

   (a) $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f : x \mapsto x^2$
   (b) $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f : x \mapsto x^2$
   (c) $f : X \rightarrow [Y \rightarrow X]$ given by $f(x) := h_x$ where $h_x : Y \rightarrow X$ is given by $h_x(y) := x$. 

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10. [6 points] Recall that $[X \to Y]$ denotes the set of functions with domain $X$ and codomain $Y$. Let $X$ and $Y$ be nonempty sets, and let $F : [X \to Y] \to [X \to (Y \times Y)]$ be given by $F(f) := h_f$, where $h_f : X \to (Y \times Y)$ is given by $h_f(x) := (f(x), f(x))$ for all $x$.

(a) Show that $F$ is injective. *Note: $g_1 = g_2$ if and only if, for all $x$, $g_1(x) = g_2(x)$."

(b) Show that $F$ is not necessarily bijective.