Lecture 9: Countability

- Countable sets
- Uncountable sets & Diagonalization
- (time permitting) relations

Last time:

$$|\mathbb{N}| = |\mathbb{N} \cup \{-1\}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}|$$

Def$^{\underline{n}}$: A set X is countable if $|X| \leq |\mathbb{N}|$
(equ. if $|\mathbb{N}| \geq |X|$)

if X is countable, I can put elts of X in a list: $X = \{x_0, x_1, x_2, \dots\}$
(define $x_0 := f(0)$, $x_1 := f(1)$, .., where f is surj from $\mathbb{N} \to X$)

e.g. $\mathbb{N} = \{\underset{n_0}{0}, \underset{n_1}{1}, \underset{n_2}{2}, \dots\}$

e.g. $X = \{\underset{x_0}{.5}, \underset{x_1}{1.5}, \underset{x_2}{2.5}, \underset{x_3}{3.5}, \dots\}$

$f : \mathbb{N} \to X$ given by $f(i) = i + 0.5$

$(0,0)^{⓪}$ $(0,1)^{②}$ $(0,2)^{⑤}$ ...
$(1,0)^{①}$ $(1,1)^{④}$
$(2,0)^{③}$
$\vdots$

e.g. $\mathbb{N} \times \mathbb{N} = \{(0,0), (1,0), (0,1),$
$(2,0), (1,1), (0,2)$
$\dots\}$

Claim: $2^{\mathbb{N}}$ is uncountable.

Proof: WTS $2^{\mathbb{N}}$ is uncountable, i.e. $|\mathbb{N}| \not\geq |2^{\mathbb{N}}|$, in other words, there is no surjection $f: \mathbb{N} \to 2^{\mathbb{N}}$. Assume (for sake of contra.), that there does exist a surj. $f: \mathbb{N} \to 2^{\mathbb{N}}$. We will show f is not a surj., giving us a contra.

For ex, $f: \mathbb{N} \to 2^{\mathbb{N}}$ might look like:

$i \in f(n)$?

n	$f(n)$	0	1	2	3	4	...
0	$\emptyset$	no	no	no	...		
1	$\mathbb{N}$	yes	yes	yes	...		
2	evens	yes	no	yes	no	yes	
3	$\{1,2,3\}$	no	yes	yes	yes	no	no no ...
4	$\emptyset$	no	no	no	no	no	
⋮							
S_D		yes	no	no	no	yes	

diff.

in general, $S_D = \{ i \mid i \notin f(i) \}$

then $S_D \neq f(k)$ for any $k \in \mathbb{N}$, because k is either (1) in $f(k)$ or (2) not in $f(k)$.

if $k \in f(k)$, then $k \notin S_D$, while if $k \notin f(k)$, then $k \in S_D$. In either case, $S_D \neq f(k)$.

So S_D is not in Image(f), a contradiction to fact that f is surj.

n	$f'(n)$	0	1	2	...
0	S_D	yes			
1	$\emptyset$		no		
2	$\mathbb{N}$			yes	
3	evens				
4	$\{1,2,3\}$				
5	$\emptyset$				
⋮	⋮				
S_D'		no	yes	no	

Claim: $\mathbb{R}$ is uncountable.

Pf: in fact, I'll show $\mathbb{R}^{[0..1)} := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ is uncountable, i.e. $|\mathbb{N}| \not\geq \mathbb{R}^{[0..1)}$. Since $|\mathbb{R}| \geq |\mathbb{R}^{[0..1)}|$, we can con conclude $|\mathbb{N}| \not\geq |\mathbb{R}|$.

Assume $\exists f : \mathbb{N} \to \mathbb{R}^{[0..1)}$ that is surjective.

e.g. f might be

n	$f(n)$	
0	0	$= 0.00000 \cdots$
1	$1/2$	$= 0.50000$
2	$\pi - 3$	$= 0.1415 \cdots$
3	$1/3$	$= 0.3333 \cdots$
$\vdots$		$\cdots 9 \cdots$
x_D		$= 0.55684 \cdots$

$+5$

In general, i^{th} digit of x_D is ith digit of $f(i) + 5$ (looping around) as shown.

Then $x_D \neq f(k)$ for any k, because
$$x_D - f(k) \geq 5 \cdot 10^{-k}$$
(or something like that...)

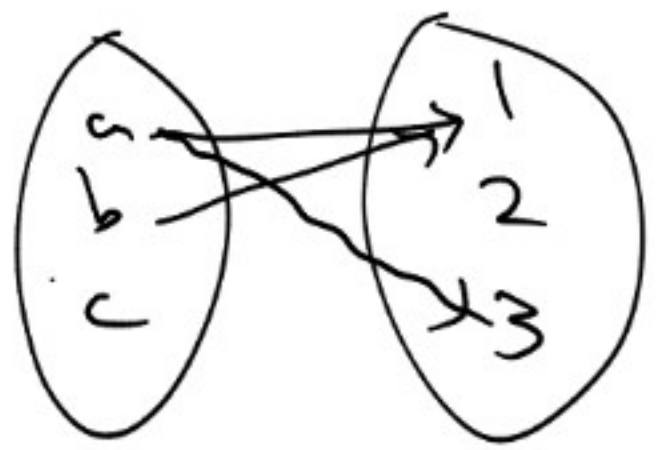
diagonalize $\mathbb{N}$?

n	$f(n)$	?
0	0	0 0 0 0 ... 0
1	1	0 0 0 1
2	2	0 0 0 ... 2
3	3	
4	4	
⋮	⋮	

Relations

- A function maps inputs to outputs, must be unambiguous (i.e. only one output for any input).

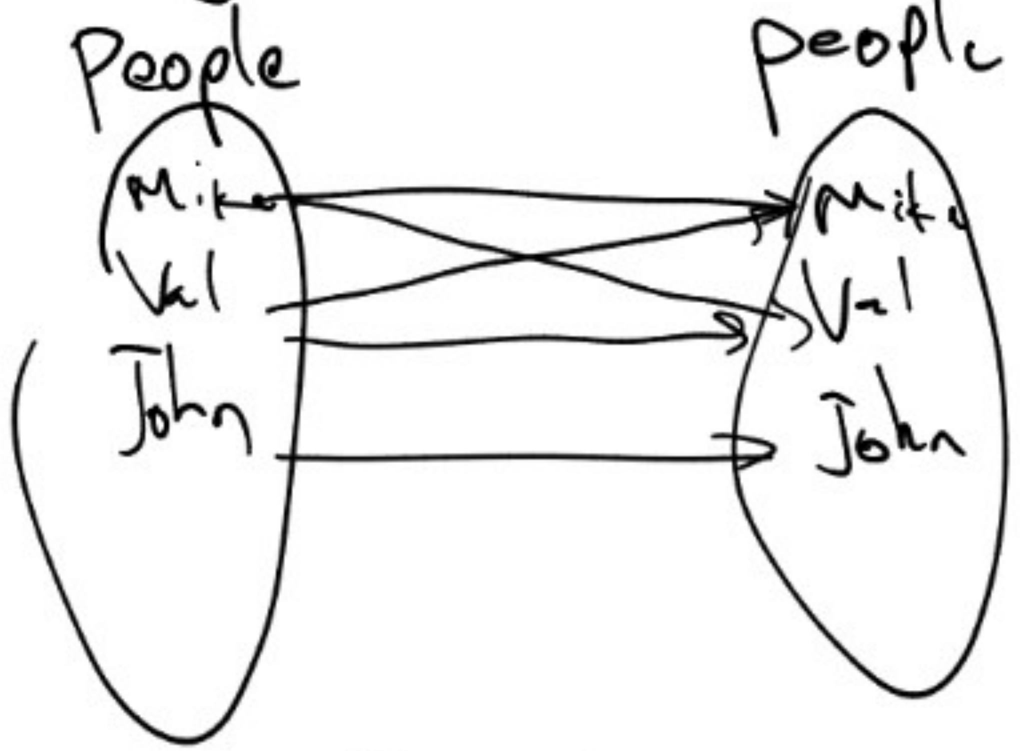
- A (binary) relation relating inputs and outputs, no additional restrictions



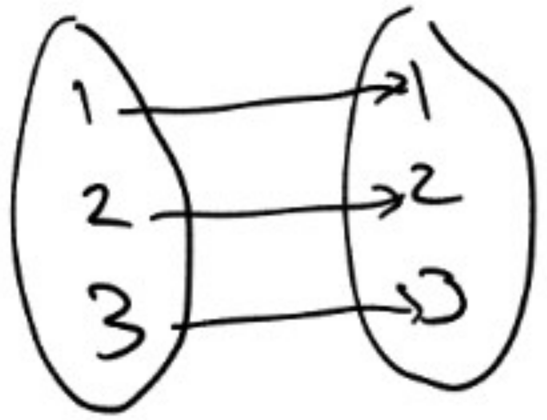
not a $f^{\underline{n}}$
is a relation: a is related to 1
a is related to 3.

Examples of relations

- Family relationships are relations



- is-a-friend
- = is a relation



(=)

- < relation