Combinatorics

Question 1: A standard license plate consists of 4 letters followed by three digits. How many possible standard license plates are there?

Question 2: A vanity plate consists of either 4, 5, or 6 characters, each of which is either a letter or a digit. How many possible vanity plates are there?

Question 3: Every license plate is either a vanity plate or a standard plate. How many license plates are there?

\[
\begin{align*}
&1. \text{ choose first char. } c_1 \quad (26 \text{ opts}) \\
&2. \text{ choose 2nd char } c_2 \\
&3. \text{ choose 3rd char } c_3 \\
&4. \text{ choose 1st digit } d_1 \quad (10 \text{ opts}) \\
&5. \text{ choose 2nd digit } d_2 \\
&6. \text{ choose 3rd digit } d_3 \\
&\text{ output sequence } (c_1, c_2, c_3, d_1, d_2, d_3)
\end{align*}
\]

\[
\begin{align*}
&36 \times 36 \times 36 \times 10 \times 10 \times 10 = 26^4 \times 10^3
\end{align*}
\]

Total: \(26^4 + 36^5 + 36^6\)
Finite cardinality definitions

Recall: $|A| = |B|$ means there exists a bijection $f : A \rightarrow B$

**Defn:** If $A$ is a set and $n \in \mathbb{N}$, then $|A|=n$ means

$$|\{1, 2, 3, \ldots, n\}| = |A|$$

i.e., there is a bijection $f : \{1, 2, \ldots, n\} \rightarrow A$.

In this case, we say $A$ is finite.

This bijection lets us write $A = \{a_1, a_2, \ldots, a_n\}$.

**Note:** If $|A| = n_1$ and $|A| = n_2$, then $n_1 = n_2$. 
Sum rule

if \( |A| = \{1, \ldots, n\} \) then \( |A| = n \)

Claim: If \( A \) and \( B \) are disjoint then \( |A \cup B| = |A| + |B| \)

Question: What does \( |A| + |B| \) mean?

Restated claim:

if \( |A| = n \) and \( |B| = m \)

then \( |A \cup B| = n + m \)

Proof:

Assume \( |A| = n \), so \( \exists f: \{1, \ldots, n\} \to \mathbb{Z} \).

A bijection \( f \) is:

\[ f(i) = \begin{cases} g(i) & \text{if } 1 \leq i \leq n \\ 2g(i - n) & \text{if } n < i \leq n - m \end{cases} \]

We shall construct \( A \cup B \), also a bijection.

\[ h(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq n \\ g(i - n) & \text{if } n < i \leq n - m \end{cases} \]

\[ 36^\circ \text{ plate (36\,')} \]

\[ 5 \text{ deg. plate (36\,')} \]

\[ 36^\circ + 36^\circ \]
Product rule

Claim: \(|A \times B| = |A| \cdot |B|\), i.e. if \(|A| = k\) and \(|B| = \ell\) then \(|A \times B| = k\ell\).

Proof 1:

\(\exists f: \exists l \ldots k \to A\) such that \(f: a \mapsto a\)

\(\exists g: \exists l \ldots \ell \to B\) such that \(g: b \mapsto b\)

\(\forall h: \exists l \ldots k \times \ell \to A \times B\) such that \(h(f(a), g(b)) = (a, b)\)
Claim: \(|A \times B| = |A| \cdot |B|\), i.e. if \(|A| = k\) and \(|B| = \ell\) then \(|A \times B| = k\ell\).

Proof 2:

1. Choose elt. of \(A\) (a)
   (\(k\) opts)

2. Choose elt. of \(B\) (b)
   (\(\ell\) opts)

Output \((a, b)\)

Total \(k\ell\) options = \(k + k + k + \ldots + k\)

\(\ell\) times

by sum rule.

\(|A \times B| = k\ell\) by arithmetic.
Counting using processes

To find $|A|$:

- Describe a process for constructing a single element of $A$ (tree diagram helps).
- Count the number of options for each choice you make.
- Use the sum rule, product rule, etc. to combine those numbers to arrive at $|A|$.
- Handle overcounting

**Note:** For this section of the course (i.e. when doing combinatorics problems), we are most interested in your description of the process and the rules you use to combine things together than we are in either a proof of the existence of a bijection or computing numbers.
The number of subsets

Suppose \(|A| = n\). How many subsets of \(A\) are there?

\[
A = \{a_1, a_2, \ldots, a_n\}
\]

To construct \(B \subseteq A\):

1. Choose \(k\), the size of \(B\).
2. Choose a subset of size \(k\) of \(A\).

\(2^n\) is the power set \(\mathcal{P}(A)\).

To construct \(B \subseteq A\):

1. Decide whether \(a_1 \in B\) (2 options)
2. Decide whether \(a_2 \in B\) (2 options)
3. ... (2 options)

\[2 \times 2 \times \ldots \times 2 = 2^n\] by product rule

Output \(B\), the set of \(a_i\)'s chosen.