## Proof outlines / proof techniques (from last time)

<table>
<thead>
<tr>
<th>Proposition</th>
<th>To prove it</th>
<th>To use it</th>
<th>To disprove</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \land Q )</td>
<td>Prove both ( P ) and ( Q )</td>
<td>Use either ( P ) or ( Q )</td>
<td></td>
</tr>
<tr>
<td>( P \lor Q )</td>
<td>Prove ( P ). Alternatively, prove ( Q )</td>
<td>Prove ( R ) in the ( P ) and ( Q ) cases to conclude ( R ) (case analysis)</td>
<td></td>
</tr>
<tr>
<td>( \forall x \in A, , P(x) )</td>
<td>Prove ( P(y) ) for an arbitrary ( y \in A )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Claim:** (for all sets \( A, \, B, \) and \( C \)) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \subseteq \)

**Proof:** Choose arbitrary sets \( A, \, B, \) and \( C. \) We want to show \( \text{LHS} \subseteq \text{RHS} \) and \( \text{RHS} \subseteq \text{LHS}. \)

To see \( \text{LHS} \subseteq \text{RHS}, \) choose an arbitrary \( x \in \text{LHS}. \) Then either \( x \in A \) or \( x \in B \cap C. \) In the former case, we have either \( x \in A \) or \( x \in B, \) so we know \( x \in A \cup B; \) Similarly, \( x \in A \cup C. \) Thus in this case, \( x \in \text{RHS}. \)

In the latter case, we have \( x \in B \cap C \) so \( x \in B \) and thus \( x \in A \cup B; \) similarly \( x \in A \cup C \) so again \( x \in \text{RHS}. \) In all cases where \( x \in \text{LHS}, \) we see \( x \in \text{RHS}. \)

To see that \( \text{RHS} \subseteq \text{LHS}, \) choose an arbitrary \( x \in \text{RHS}. \) We know that either \( x \in A \) or \( x \notin A. \) In the former case, since \( x \in A, \) we have \( x \in A \cup (B \cap C) = \text{LHS}. \)

In the latter case, we know that either \( x \in A \) or \( x \in B \) (since \( x \in A \cup B), \) but since \( x \notin A \) we must have \( x \in B. \) A similar argument shows \( x \in C, \) so \( x \in B \cap C \) and thus \( x \in \text{LHS}. \)
Injectivity

\[ f : A \rightarrow B \text{ is injective if for all } x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]

\[ g : A \rightarrow B \text{ is not injective because } g(1) = g(2), \text{ i.e., there are 2 inputs giving the same output.} \]

\[ \text{Def: } f \mid A \rightarrow B \text{ is injective if } \forall x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]

\[ \text{All: } f : A \rightarrow B \text{ is inj. if } \forall x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]

\[ \text{Alt: } f : A \rightarrow B \text{ is inj. if } \forall x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]

Contrapositives:

- "if P then Q" is contrapositive of "if Q is false then P is false".
- "if Q is false then P is false" is contrapositive of "if P then Q".

"Official def":

\[ \forall x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]
Composition and inverses

\[ \begin{array}{ccc}
A & \rightarrow & B \\
1 & \rightarrow & a \\
2 & \rightarrow & b \\
3 & \rightarrow & c \\
\end{array} \]

\[ \begin{array}{ccc}
B & \rightarrow & C \\
a & \rightarrow & y \\
b & \rightarrow & z \\
\end{array} \]

**Definition:** If \( f : A \rightarrow B \) and \( g : B \rightarrow C \), then \( (g \circ f) : A \rightarrow C \) is given by

\[ (g \circ f)(a) := g(f(a)) \]

**Definition:** We say that \( g \) is a left-inverse of \( f : A \rightarrow B \) if for all \( x \in A \),

\[ g(f(x)) = x \]

i.e., \( (g \circ f)(x) = x \).