**Instructions:** This is a 150 minute exam. Please answer the following questions in the provided booklet. Ensure that your name and netid are on your exam booklet. Clearly indicate your answer to each question. Books, notes, calculators, laptops, and carrier pigeons are all disallowed. You may leave mathematical expressions unevaluated (e.g. just write $17 \cdot 3$ instead of 51).

1. Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.
   (a) The set $\Sigma^*$ containing all finite length strings of 0’s and 1’s.
   (b) The set $2^\mathbb{N}$ containing all sets of natural numbers.
   (c) The set of all Turing machines.
   (d) The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.
   (e) The set $[\mathbb{N} \to \{0,1\}]$ containing all functions from $\mathbb{N}$ to $\{0,1\}$.

   Be sure to include enough detail:
   • If listing elements, be sure to clearly state how you are listing them;
   • If diagonalizing, be sure it is clear what your diagonal construction is;
   • If providing a function, make sure it is clear what the output is on a given input.

2. Consider a disease in which one out of a hundred people has. Given a test that will answer “yes” with probability $999/1000$ if a person has the disease and will answer “yes” with probability $2/1000$ if the person does not have the disease. If the test says “yes” what is the probability that the person has the disease?

3. (a) Give the definition of variance in terms of expectation.
   (b) Let $X$ and $Y$ be random variables with $E(X) = E(Y) = 0$. Prove that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. Make (and clearly state) additional assumptions if necessary.

4. Suppose you are given a function $f : \mathbb{N} \to \mathbb{N}$, and are told that $f(1) = 1$ and for all $n$, $f(n) \leq 2f(\lfloor n/2 \rfloor) + 1$.
   Use strong induction on $n$ to prove that for all $n \geq 2$, $f(n) \leq 2n \log_2 n$.

   You may write log to indicate $\log_2$. Here is a reminder of some facts about $|x|$ and $\log x$:
   • $|x| \leq x$
   • $\log 1 = 0$, $\log 2 = 1$
   • $\log(x/2) = \log x - 1$
   • $\log(2^x) = x$
   • $\log(x^2) = 2 \log x$
   • If $x \leq y$ then $\log x \leq \log y$
5. Convert the nondeterministic finite automata defined by the table below to a deterministic finite automata. A is the start state; The set of final states is \(\{C, D, F\}\).

\[
\begin{array}{c|cc}
\delta & 0 & 1 \\
\hline
A & \{B, C\} & \emptyset \\
B & \emptyset & \{C\} \\
C & \{B\} & \emptyset \\
D & \emptyset & \{E\} \\
E & \{F\} & \emptyset \\
F & \{D\} & \emptyset \\
\end{array}
\]

6. Use the pumping lemma to prove that the set \(\{a^ib^j \mid i < j\}\) is not regular.

7. Define the halting problem to be the set \(L_H = \{(M, x) \mid M \text{ halts when started on input } x\}\).

   (a) Is \(L_H\) recursively enumerable? Give a 3-5 sentence justification. If building a machine, be sure to describe the algorithm that the machine uses.

   (b) Is \(L_H\) recursive? Give a 3-5 sentence justification. If building a machine, be sure to describe the algorithm that the machine uses.

8. Use Euler’s theorem and repeated squaring to efficiently compute \(8^n \mod 15\) for \(n = 5\), \(n = 81\) and \(n = 16023\). Hint: you can solve this problem with 4 multiplications of single digit numbers. Please fully evaluate all expressions for this question (e.g. write 15 instead of \(3 \cdot 5\)).