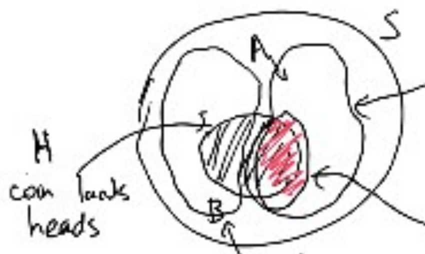
Lecture 32: Conditional probability

- Examples / Def$\underline{^n}$s
- Prob. trees
- Bayes' rule, law of total probability

Example: 1^{st} choose coin a or coin b, then flip.

- coin a lands heads w/ pr $3/4$ $\{$ Interpret: $Pr(H|A) = 3/4$.
- coin b " " pr $1/2$. $\{$ Interpret $Pr(H|B) = 1/2$
- coin a chosen w/ pr $1/3$.

Reasonable sample space: $\{(a,h),(a,t),(b,h),(b,t)\} = S$

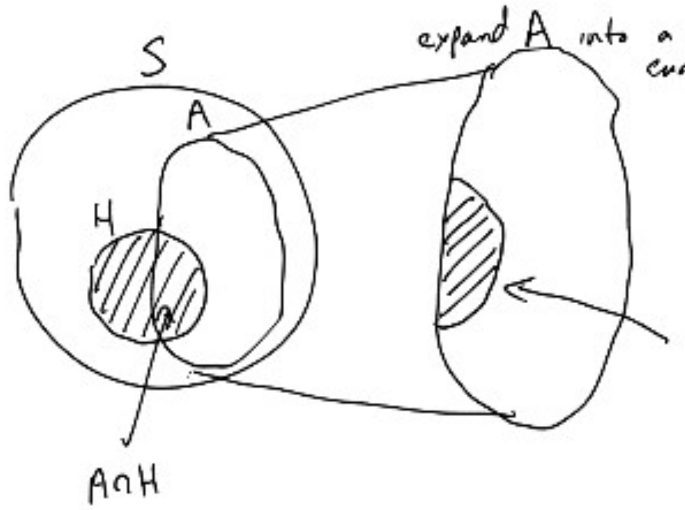


choose coin a. $Pr(A) = 1/3$.

$A = \{(a,h),(a,t)\}$

$\boxed{\{(a,h)\}}$ "coin a lands heads w pr $= 3/4$" doesn't mean $Pr(A \cap H) = 3/4$

choose coin B, $B = S \setminus A$, $Pr(B) = 1 - 1/3 = 2/3$



Prob. of H, "given" A.

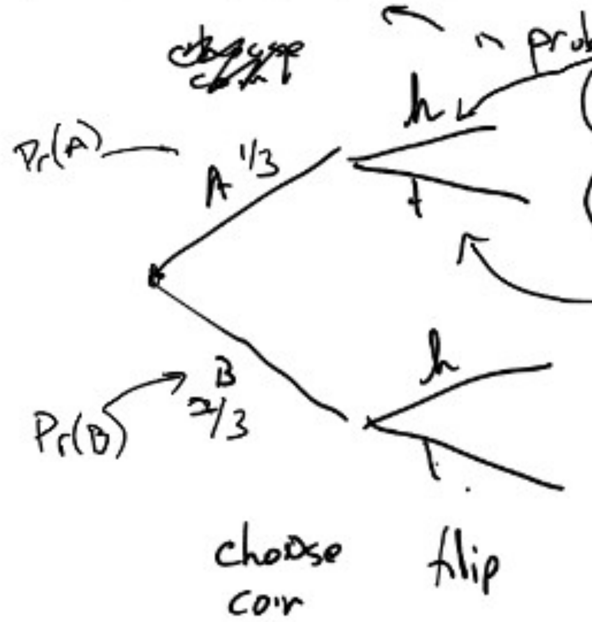
$$\frac{Pr(A \cap H)}{Pr(A)} = \underbrace{Pr(H|A)}_{\text{conditional probability}}$$

scaling up so that $Pr(A) = 1$.

$$Pr(H|A) = \frac{Pr(H \cap A)}{Pr(A)}$$

Know: $Pr(A) = 1/3$ $\quad Pr(B) = 2/3$ $\quad Pr(H|A) = 3/4$ $\quad Pr(H|B) = 1/2$

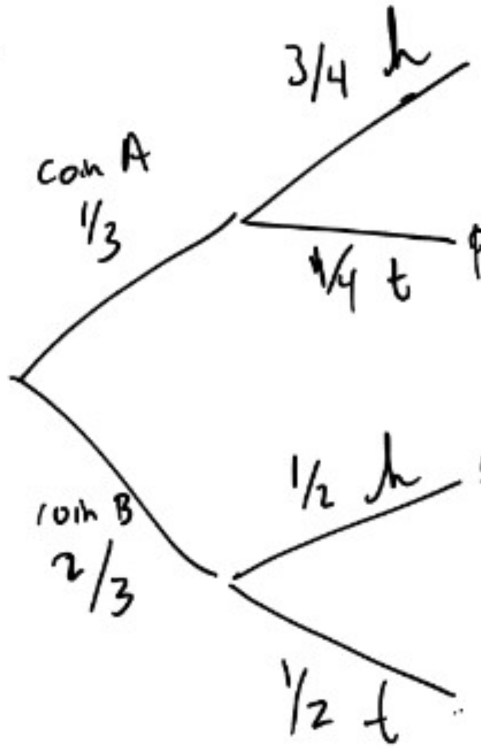
in prob calculated in prob.



$Pr(H|A) = 3/4$.

$Pr(T|A) = 1 - Pr(H|A) = 1/4$

to be prove.



$(a,h) = A \cap H$

$Pr(A \cap H) = Pr(H|A) \cdot Pr(A)$.

$Pr(\{(a,t)\}) = 1/12$

$Pr = 1/3$

$Pr = 1/3$.

$$Pr(A \cap H) = Pr(\{(a,h)\}) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

$$Pr(H) = Pr(A \cap H) + Pr(B \cap H) = \frac{1}{3} + \frac{1}{4}$$

disjoint.

Useful facts about cond. prob.

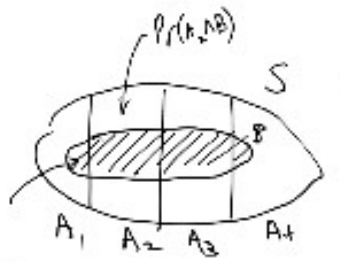
Bayes' rule: $\Pr(A|B) = \dfrac{\Pr(B|A)\Pr(A)}{\Pr(B)}$

Proof: $\Pr(A|B) = \dfrac{\Pr(A\cap B)}{\Pr(B)}$ plug in $\Pr(B|A) = \dfrac{\Pr(A\cap B)}{\Pr(A)}$

$\Pr(A)\Pr(B|A) = \Pr(A\cap B)$

$\Pr(A|B) = \dfrac{\Pr(B|A)\Pr(A)}{\Pr(B)}$ ✓

Law of total prob.



$\Pr(A_1\cap B) = \Pr(B|A_1)\cdot\Pr(A_1)$

A_i partition S (disjoint cover)

$\Pr(B)$?

$= \sum \Pr(A_i\cap B)$

$= \sum_i \Pr(B|A_i)\Pr(A_i)$

$= \Pr(B|A_1)\Pr(A_1) + \dots \Pr(B|A_n)\Pr(A_n)$

in prev. ex.

$\Pr(H) = \Pr(H|A)\cdot\Pr(A) + \Pr(H|B)\cdot\Pr(B)$

A, B partition S

Ex: Disease occurs with Pr. $\frac{1}{10{,}000}$

Test: false positive rate $\left(\Pr(\text{test pos} \mid \text{don't have disease})\right) = \frac{1}{100}$

false negative rate $\left(\Pr(\text{test neg} \mid \text{have ds})\right) = \frac{5}{100}$

Take test, test is positive!

P = test is positive. N = test neg.

D = have disease H = healthy.

$\Pr(D) = \frac{1}{10{,}000}$

$\Pr(H) = \frac{9999}{10{,}000}$

$\Pr(P|H) = 1/100$

$\Pr(N|D) = 5/100$

$\Pr(P|D) = 95/100$

$$\Pr(D|P) = \frac{\Pr(P|D)\cdot\Pr(D)}{\Pr(P)} = \frac{\Pr(P|D)\Pr(D)}{\Pr(P|D)\cdot\Pr(D) + \Pr(P|H)\cdot\Pr(H)}$$

Bayes'

$\frac{95}{100}\cdot\frac{1}{10{,}000}$ very small.

$\sim \frac{1}{10{,}000}$

$1/100 \cdot \frac{9999}{10{,}000} \approx \frac{1}{100}$

$$\sim \frac{\frac{1}{10{,}000}}{\frac{1}{10{,}000} + \frac{1}{100}}$$ small