Order of quantifiers

What do the following mean?

- (a) "$\forall x$ ($\exists y$ s.t. x loves y)" means everybody loves somebody
- (b) "$\exists y$ (s.t. $\forall x$, x loves y)" means there is somebody who is loved by everybody.

$\exists x$ s.t. $\forall y$ x loves y. somebody loves everybody?

pf of (a): choose arb. x, let $y = x$'s mother. WTS x loves y.
well, y gave birth to x, so x must love y.

pf of (b): ~~let $y = x$'s mother~~
let y = raymond. Choose arb. x. Then x loves Raymond because everybody loves Raymond. ✓

Negating quantified statements

- **Defn:** $|A| \geq |B|$ if there is a function $f : A \to B$ that is surjective
- **Defn:** $f : A \to B$ is surjective if every $y \in B$, y is in the image of f
- **Defn:** y is in the image of f if there is some $x \in A$ with $f(x) = y$

Putting these together (symbolically): $|A| \geq |B|$ if:

$$\neg \left(\exists f : A \to B \left(\forall y \in B, \left(\exists x \in A \text{ st. } f(x) = y \right) \right) \right)$$

means

$\forall f, \neg(\forall y (\dots))$

$\forall f, \exists y, \neg(\dots)$

Question: what does $|A| \not\geq |B|$ mean?

- in this style
- in this style

Reminder: $\forall x, P(x)$ is false if $\underbrace{\exists x, \overbrace{P(x)}^{\text{"not" } \neg P(x)} \text{ is false}}$

$\exists x, P(x)$ is false if $\forall x, \neg P(x)$

Answer:

- $|A| \not\geq |B|$ means there is not $f : A \to B$ is surjective.
  i.e. $\forall f : A \to B$, f is not surjective.
- f is not surjective if it is false that every y is in image of f
  i.e. $\exists y \in B$ that is not in im. of f.
- y is not in img of f means that there is no x that maps to y,
  i.e. $\forall x \in A$, $f(x) \neq y$.

So $|A| \not\geq |B|$ means

$$\forall f : A \to B, \exists y \in B, \forall x \in A, f(x) \neq y.$$

Relations

function: table

x	$f(x)$
0	a
1	b
2	c
$\vdots$	$\vdots$

relation

Name	Salary	Title	Emp. Date ...
Mike	a lot	A	2013
Mike	a little	B	2015

set of rows, each row has Name, Salary, ...

$$\left\{ \begin{array}{l} (\text{Mike}, \text{a lot}, A, 2013) \\ (\text{Mike}, \text{a little}, B, 2015) \\ \quad \vdots \end{array} \right\}$$

$\subseteq$ Names $\times$ Salaries $\times$ Titles $\times$ Dates.

Def$^{\underline{n}}$: A relation R on sets $A, B, C, \ldots$ is a subset of $A \times B \times C \times \cdots$

Binary relations

Def$\underline{n}$: A binary relation on a set A is a subset of $\underbrace{A \times A}_{\text{ordered pairs}}$.

Example: is-friend-of is a relation on the set of people.

$\{(\text{Alice}, \text{Bob}), (\text{Bob}, \text{Alice}), (\text{Bob}, \text{Chuck}), (\text{Chuck}, \text{Dave})\}$

bob is a friend of chuck
chuck is not a friend of bob.

Notation: if R is a bin. rel$\underline{n}$ on A, we write xRy to mean $(x,y) \in R$.

Example: "=" is a relation on any set.

like "=" for f$\underline{n}$s is a relation on set of all f$\underline{n}$s.

"=" for sets is " " sets.

"$|\cdot| = |\cdot|$" is a relation on set of all sets.

"$\neq$" is a binary rel$\underline{n}$.

"$\leq$" $3 \leq 5$ means $(3,5) \in$ "$\leq$"

Properties of "equivalence"

we expect "equivalence" R on a set A to satisfy:

- Reflexivity: $\forall x \in A,\ x R x$
- Symmetry: $\forall x, y \in A$, if $x R y$ then $y R x$.
- Transitivity: $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x R z$.