Discrete Structures	*Prelim 2 sample questions— **Solutions***
CS2800	*Questions selected for Spring 2019*

1. *Prove that $7^m - 1$ is divisible by 6 for all positive integers m (try this both inductively and using equivalence classes).*

   Solution There are two ways to do this. One way: notice that $7 \equiv 1 \mod 6$, thus $7^m \equiv 1 \mod 6$ for any m (applying the known result that "if $a \equiv b \pmod m$ and $c \equiv d \pmod m$ then $ac \equiv bd \pmod m$" $m-1$ times), and thus $7^m - 1 \equiv 0 \mod 6$. This implies $7^m - 1$ is divisible by 6.

   Alternatively you can do a direct proof by induction:

   Base case: $m = 1$, $7^1 - 1 = 6$ which is obviously divisible by 6.

   Inductive step: Assume $7^m - 1$ is divisible by 6 for some $m \geq 1$ (inductive hypothesis). Then $7^{m+1} - 1 = 7^{m+1} - 7 + 6 = 7(7^m - 1) + 6$. But $7^m - 1$ is divisible by 6 (by the inductive hypothesis) and so is 6, so $7^{m+1} - 1$ is also divisible by 6. Hence proved by induction.

2. *[6 points]* Pascal's triangle *is a sequence of rows, where each entry is formed by adding the two adjacent entries from the previous row:*

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ \cdots \end{array}$$

   If we let $P_{n,k}$ stand for the kth entry in the nth row of Pascal's triangle, then $P_{n,k}$ is given by the formulas $P_{1,1} ::= 1$, $P_{n,0} ::= 0$ for all n, and $P_{n+1,k} ::= P_{n,k-1} + P_{n,k}$ if $n \geq 1$.

   Prove by induction on n that for all $n \geq 1$, for all k with $1 \leq k \leq n$, $P_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

   Note: *The definition of $n!$ is $0! ::= 1$ and $n! ::= n \cdot (n-1)!$ for all $n \geq 1$.*

   Solution Proof by induction. Let $P(n)$ be the statement $\forall k \in [1..n], P_{n,k} = \binom{n}{k}$.

   $P(1)$ is true, because the only $k \in [1..1]$ is $k = 1$, and $P_{1,1} = 1$ and $\binom{1}{1} = 1!/0!1! = 1$.

   Now, assume $P(n)$; we wish to show $P(n+1)$. Choose an arbitrary k. We have

$$\begin{aligned}
P_{n+1,k} &= P_{n,k-1} + P_{n,k} && \text{by definition} \\
&= \binom{n}{k-1} + \binom{n}{k} && \text{by induction hypothesis} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} && \text{by definition} \\
&= n!\frac{k + (n-k+1)}{k!(n-k+1)!} && \text{putting things over a common denominator} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} && \text{algebra} \\
&= \binom{n+1}{k} && \text{by definition}
\end{aligned}$$

   as required.

3. *Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$*

Solution Base case: when $n = 0$, the left hand side is $2^0 = 1$ and the right hand side is $2^2 - 1 = 1$, and they are clearly the same.

Inductive step: Choose an arbitrary n and assume that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (this is the inductive hypothesis).

We wish to show that $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$. We compute:

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} && \text{arithmetic} \\
&= (2^{n+1} - 1) + 2^{n+1} && \text{by the inductive hypothesis} \\
&= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1
\end{aligned}$$

as required.

4. *The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:*

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \quad \textit{for } n \geq 2
\end{aligned}$$

That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers n (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

Solution Let $P(n)$ be the statement "$\sum_{i=0}^{n} F_i = F_{n+2} - 1$. We must show $P(0)$ and $P(n+1)$ assuming $P(n)$.

To see $P(0)$, note that $\sum_{i=0}^{0} F_i = F_0 = 1$, while $F_{0+2} - 1 = F_0 + F_1 - 1 = 1 + 1 - 1 = 1$. Since they are the same, $P(0)$ holds.

To see $P(n+1)$, first assume $P(n)$. We have

$$\begin{aligned}
\sum_{i=0}^{n+1} F_i &= \sum_{i=0}^{n} F_i + F_{n+1} \\
&= F_{n+2} - 1 + F_{n+1} && \text{by } P(n) \\
&= F_{n+1+2} - 1 && \text{by definition of } F_{n+1+2}
\end{aligned}$$

as required.

5. *Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.*

Solution Let $P(n)$ be the statement "$n^2 - 7n + 12$ is non-negative." We must show $P(3)$, and for any $n \geq 3$, $P(n+1)$ assuming $P(n)$.

To see $P(3)$, note that $3^2 - 7 \cdot 3 + 12 = 0 \geq 0$.

Now, assume $n \geq 3$ and $P(n)$; we want to show $P(n+1)$. Well,

$$\begin{aligned}(n+1)^2 - 7(n+1) + 12 &= n^2 + 2n + 1 - 7n - 7 + 12 \\ &= (n^2 - 7n + 12) + (2n + 1 - 7) \\ &\geq 2n + 1 - 7 && \text{by } P(n) \\ &\geq 0 && \text{since } n \geq 3\end{aligned}$$

6. *Chapter 5 of MCS has a bunch of good induction exercises (and you can find even more by searching)*

7. *Suppose that Alice sends the message a to Bob, encrypted using RSA. Suppose that Bob's implementation of RSA is buggy, and computes $k^{-1} \mod 4\phi(m)$ instead of $k^{-1} \mod \phi(m)$. What decrypted message does Bob see? Justify your answer.*

Solution Alice transmits $a^k \mod m$ to Bob, who then computes $(a^k)^{k^{-1}} \mod m$. Because Bob miscomputed k^{-1}, we know that $kk^{-1} \equiv 1 \mod 4\phi(m)$. In other words, $kk^{-1} = 1 + t \cdot 4\phi(m)$ for some t. Therefore Bob receives

$$\begin{aligned}(a^k)^{k^{-1}} &\equiv a^{1+4t\phi(m)} \\ &\equiv a \cdot a^{4t\phi(m)} \\ &\equiv a \cdot (a^{\phi(m)})^{4t} \\ &\equiv a \cdot 1^{4t} \\ &\equiv a \mod m\end{aligned}$$

8. *(a) What are the units of $\mathbb{Z}$ mod 12?*

Solution A unit in a set of numbers is a number that has an inverse. In the set $\mathbb{Z}_{12} = \{[0], [1], [2], [3], [4], [5], [6], [7], [8]$ the units are [1], [5], [7], and [11]. In general, $[n]$ is a unit mod m if n and m are relatively prime.

(b) What are their inverses?

Solution $[1]^{-1} = [1]$, $[5]^{-1} = [5]$, $[7]^{-1} = [7]$, and $[11]^{-1} = [11]$. This is because $[1] \cdot [1] = [1]$, $[5] \cdot [5] = [25] = [1]$, $[7] \cdot [7] = [49] = [1]$ and $[11] \cdot [11] = [121] = 1$.

(c) What is $\phi(12)$?

Solution By definition of ϕ, $\phi(12)$ is the number of units mod 12. Since there are 4 units, $\phi(12) = 4$.

9. *Use Euler's theorem and repeated squaring to efficiently compute $8^n \mod 15$ for $n = 5$, $n = 81$ and $n = 16023$. Hint: you can solve this problem with 4 multiplications of single digit numbers. Please fully evaluate all expressions for this question (e.g. write 15 instead of $3 \cdot 5$).*

Solution We use the fact that $8^{\varphi(15)} = 1 \mod 15$. $\varphi(15) = (3-1)(5-1) = 8$ [multiplication #1], so we can reduce all of the exponents mod 8. We then use repeated squaring to compute 8^{2^k}:

$$\begin{aligned}
&[8]^1 = [8] \\
&[8]^2 = [64] = [4] && \text{[multiplication \#2]} \\
&[8]^4 = [4]^2 = [16] = [1] && \text{[multiplication \#3]}
\end{aligned}$$

We can then use these to compute the powers of [8]:

$$\begin{aligned}
[8]^5 &= [8]^4[8] = [1][8] = [8] \\
[8]^{81} &= [8]^1 = [8] \\
[8]^{16023} &= [8]^7 = [8]^4[8]^2[8] = [1][4][8] = [32] = [2] && \text{[multiplication \#4]}
\end{aligned}$$

10. *In this problem, we are working mod 7, i.e. $\equiv$ denotes congruence mod 7 and $[a]$ is the equivalence class of a mod 7.*

    (a) *What are the units of $\mathbb{Z}_7$? What are their inverses?*

    Solution

    - [1]'s inverse is [1]
    - [2]'s inverse is [4]
    - [3]'s inverse is [5]
    - [4]'s inverse is [2]
    - [5]'s inverse is [3]
    - [6]'s inverse is [6]

    (b) *Compute $[2]^{393}$.*

    Solution $[2]^{393} = ([2]^3)^{131} = [1]^{131} = [1]$

11. (a) *Recall Bézout's identity from the homework: for any integers n and m, there exist integers s and t such that $gcd(n,m) = sn + tm$. Use this to show that if $gcd(k,m) = 1$ then $[k]$ is a unit of $\mathbb{Z}_m$.*

    Solution If $gcd(k,m) = 1$ then $1 = sk + tm$. Reducing this equation mod m gives $[1] = [s][k] + [t][0] = [s][k]$. Therefore, $[k]$ has an inverse (namely $[s]$); and is thus a unit.

    (b) *Use part (a) to show that if p is prime, then $\phi(p) = p - 1$.*

    Solution Since p is prime, everything less than p is relatively prime to p, except for 0. There are $p-1$ such numbers, and thus $p-1$ units.

    (c) *Use Euler's theorem to compute $3^{38} \mod 37$ (note: 37 is prime).*

Solution $\varphi(37) = 36$, so $[3]^{38} = [3]^2 = [9] \bmod 37$.

12. *Bob the Bomber wishes to receive encrypted messages from Alice the Accomplice. He generates a public key pair $m = 21$ and $k = 5$. Luckily, you have access to an NSA supercomputer that was able to factor 21 into $7 \cdot 3$.*

    (a) *Use this information to find the decryption key k^{-1}.*

    Solution We must find the inverse of 5 mod $\phi(m) = \phi(7 \cdot 3) = (7-1)(3-1) = 12$. Experimentally, $[5 \cdot 5] = [25] = [1]$. Alternatively, you can use the pulverizer. This results in $1 = -2 \cdot 12 + 5 \cdot 5$, giving an inverse of 5.

    (b) *Without changing m, what other possible keys k could Bob have chosen? Find the decryption keys for those keys as well.*

    Solution By inspection, the units of $\mathbb{Z}_{12}$ are [1], [5], [7], and [11] (all other numbers share a factor with 12). Experimentally, they are all their own inverses. Note that [1] is not a smart key choice, but we accepted it.

    (c) *Alice encrypts a secret message msg using Bob's public key ($k = 5$), and sends the ciphertext $c = 4$. What was the original message?*

    Solution We must compute $[4]^{[5]} = [4^5]$. We see $[4^2] = [16]$; squaring this gives $[4^4] = [(4^2)^2] = [256] = [4]_{21}$. Thus $[4^5] = [4 \cdot 4^4] = [16]$.

13. *Which of the following does RSA depend on? Explain your answer briefly.*

    (a) *Factoring is easy and testing primality is hard.*

    (b) *Factoring is hard and testing primality is easy.*

    (c) *Both factoring and testing primality are hard.*

    (d) *Both factoring and testing primality are easy.*

    Solution RSA depends on (b), that factoring is hard and testing primality is easy. The public key in RSA is the product of two large primes. We couldn't generate public keys if testing primality wasn't easy. On the other hand, we could easily decrypt encrypted messages if factoring was easy (because in that case, given a public key, which is the product of two large primes could easily compute the secret – the two factors, and that's what we need to know to decrypt.

14. (a) *Let m and n be integers greater than 1. Show that the function $f : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m$ given by $f : ([a]_m, [b]_n) \to [a + b]_m$ is not necessarily well defined. [Hint: you just need an example here.]*

    Solution Consider $m = 2$, $n = 3$. Then $[1]_m = [3]_m$ and $[1]_n = [4]_n$, but $[1 + 3]_m = [0]_m$ while $[3 + 4]_m = [1]_m$.

    (b) *Show that f is well defined if $m|n$.*

    Solution Suppose $m|n$. Then $n = mc$ for some c. Suppose also that $[a]_m = [a']_m$ (so that $a = a' + md$ for some d) and $[b]_n = [b']_n$ (so that $b = b' + ne$).

    Then

    $$a + b = (a' + md) + (b' + ne) = a' + b' + md + mce = a' + b' + m(d + ce)$$

    Thus $[a + b]_m = [a' + b']_m$.

15. *We define a set S of functions from $\mathbb{Z}$ to $\mathbb{Z}$ inductively as follows:*

Rule 1. *For any $n \in \mathbb{Z}$, the translation (or offset) function $t_n : x \mapsto x + n$ is in S.*

Rule 2. *For any $k \neq 0 \in \mathbb{Z}$, the scaling function $r_k : x \mapsto kx$ is in S.*

Rule 3. *If f and g are elements of S, then the composition $f \circ g \in S$.*

Rule 4. *If $f \in S$ and f has a right inverse g, then g is also in S.*

In other words, S consists of functions that translate and scale integers, and compositions and right inverses thereof.

> ***Note:*** *This semester, we made a bigger distinction between the elements of an inductively defined set and the meaning of an inductively defined set. We probably would have phrased this question as follows: Let S be given by*
>
> $$s \in S ::= t_n \mid r_k \mid s_1 \circ s_2 \mid rinv\ s$$
>
> *and inductively, let the function defined by s (written $F_s : \mathbb{Z} \to \mathbb{Z}$) be given by the rules $F_{t_n}(x) ::= x + n$, $F_{r_k}(x) ::= ks$, $F_{s_1 \circ s_2}(x) ::= F_{s_1} \circ F_{s_2}$ and let $F_{rinv\ s} ::= g$ where g is a right inverse of F_s.*

(a) *[1 point] Show that the function $f : x \mapsto 3x + 17$ is in S.*

Solution By rule 1, the function $t_{17} : x \mapsto x + 17$ is in S, and by rule 2, $r_3 : x \mapsto 3x$ is in S. By rule 3, therefore, $t_{17} \circ r_3 : x \mapsto 3x + 17$ is in S.

(b) *Use structural induction to prove that for all $f \in S$, f is injective. You may use without proof the fact that the composition of injective functions is injective.*

Solution We must show that all functions formed with each of the rules are injective. Let $P(s)$ be the statement s is injective.

$P(t_k)$ holds, because t_k has a two sided inverse t_{-k}, and is therefore injective.

$P(r_k)$ holds, because we required that $k \neq 0$. Therefore, if $kx_1 = kx_2$, we can cancel k to find $x_1 = x_2$.

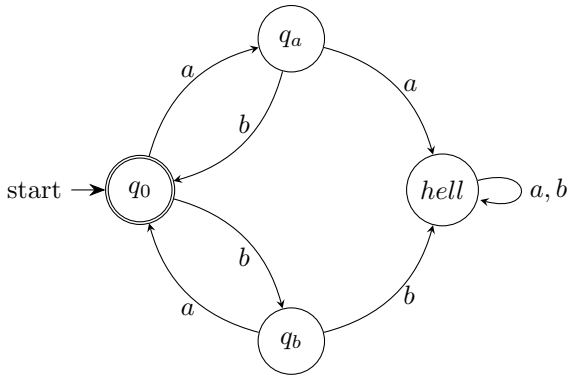
$P(f \circ g)$ holds, assuming $P(f)$ and $P(g)$, because the composition of injections is an injection.

If g is the right inverse of f, then $P(g)$ holds, because g has a left-inverse (namely f) and is therefore injective.

(c) *Give a surjection ϕ from S to $\mathbb{Z}$ (proof of surjectivity not necessary). Remember that this surjection must map a* function *to an* integer*, and for every integer there must be a function that maps to it.*

Solution Let $\phi(s) ::= s(0)$. This is a surjection, because $t_n(0) = 0 + n = n$, so for any n there exists $s \in S$ (namely t_n) with $\phi(s) = n$.

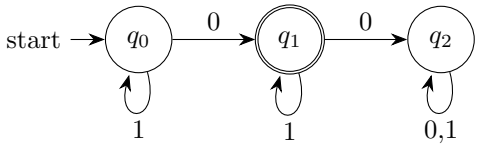
16. *Draw a finite automaton (DFA, NFA or ϵ-NFA) with alphabet $\{a, b\}$ to recognize strings of the form $x_1x_2x_3\cdots$ where each x_i is either "ab" or "ba".*



Solution

17. Build a deterministic finite automaton that recognizes the set of strings of 0's and 1's, that only contain a single 0 (and any number of 1's). Describe the set of strings that lead to each state.

Solution



The strings leading to q_i contain i 0's (2 or more for q_2).

18. (a) In lecture, we proved that if $[a]_m$ is a unit, then $[a]_m^{\varphi(m)} = [1]$.

Use this to show that $a^{[b]_{\varphi(m)}} := [a^b]_m$ is well defined.

Solution Suppose $[b] = [b']$. Then $b = b' + km$ for some k. Therefore,

$$\begin{aligned}
[a^b]_m &= [a^{b'+k\varphi(m)}]_m && \text{plugging in } b = b' + km \\
&= [a^{b'}(a^{\varphi(m)})^k] && \text{algebra} \\
&= [a^{b'}]([a]^{\varphi(m)})^k && \text{definition of modular multiplication} \\
&= [a^{b'}][1]^k && \text{by assumption} \\
&= [a^{b'}] && \text{algebra}
\end{aligned}$$

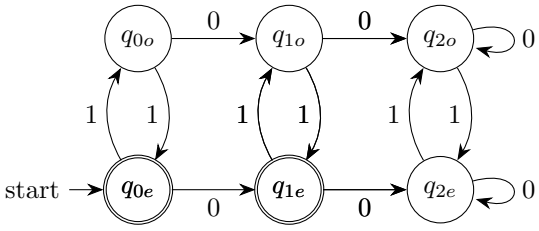
(b) Use Euler's theorem to prove that if p is prime, then $[a]_p^p = [a]_p$ (whether $[a]_p$ is a unit or not).

Solution Since p is prime, we have $\varphi(p) = p - 1$. If $[a]$ is a unit, then Euler's theorem says that $[a]^{p-1} = [1]$; multiplying both sides by $[a]$ gives $[a]^p = [a]$.

If $[a]$ is not a unit, then $gcd(a, p) \neq 1$; this means that a and p share a common factor, which can only happen if a is a multiple of p. In this case, $[a]_p = [0]$, and $[0]^p = [0]$.

*19. Give a DFA that accepts strings in $\{0, 1\}^*$ if and only if they contain at most one 0 **and** an even number of 1's. For each state, describe the strings that reach that state.*

Solution



If $\hat{\delta}(q_{e0}, x) = q_{0e}$ then x has no 0's and an even number of 1's.
If $\hat{\delta}(q_{e0}, x) = q_{0o}$ then x has no 0's and an odd number of 1's.
If $\hat{\delta}(q_{e0}, x) = q_{1e}$ then x has one 0 and an even number of 1's.
If $\hat{\delta}(q_{e0}, x) = q_{1o}$ then x has one 0 and an odd number of 1's.
If $\hat{\delta}(q_{e0}, x) = q_{2e}$ then x has two or more 0's and an even number of 1's.
If $\hat{\delta}(q_{e0}, x) = q_{2o}$ then x has two or more 0's and an odd number of 1's.

20. In this question we formalize the usual algorithm for adding numbers represented in base b. Let $\Sigma = \{0, 1, \ldots, b-1\}$.

(a) Give an inductive *definition of the base b interpretation function $n : \Sigma^* \to \mathbb{N}$. Check that if $b = 2$ then $n(110) = 6$.*

Solution $n(\varepsilon) := 0$ and $n(xa) := bn(x) + a$. If $b = 2$ we have

$$\begin{aligned} n(101) &= 2n(11) + 0 \\ &= 2(2n(1) + 1) \\ &= 2(2(2n(\varepsilon) + 1) + 1) \\ &= 2(2(0 + 1) + 1) \\ &= 2(3) = 6 \end{aligned}$$

(b) Consider the following inductive definition of a function

$$add : \Sigma^* \times \Sigma^* \times \{0, 1\} \to \Sigma^*$$

given by

$$\begin{aligned} add(\varepsilon, \varepsilon, c) &:= \varepsilon c \\ add(xd, \varepsilon, c) &:= add(x, \varepsilon, q)r \qquad \text{where } q = quot(d + c, b) \text{ and } r = rem(d + c, b) \\ add(\varepsilon, xd, c) &:= add(x, \varepsilon, q)r \qquad \text{where } q = quot(d + c, b) \text{ and } r = rem(d + c, b) \\ add(xd, ye, c) &:= add(x, y, q)r \qquad \text{where } q = quot(d + e + c, b) \text{ and } r = rem(d + e + c, b) \end{aligned}$$

Note: you know this algorithm well; c stands for "carry".

Prove that $n(add(x, y, c)) = n(x) + n(y) + c$. If multiple cases are substantially similar, you may say so instead of repeating the proof.

Solution Let $P(x, y)$ be the statement $n(add(x, y, c)) = n(x) + n(y) + c$.

We must show $P(\varepsilon, \varepsilon)$, $P(\varepsilon, xa)$, $P(xa, \varepsilon)$ and $P(xa, yb)$.

In the $P(\varepsilon, \varepsilon)$ case, we have $n(add(\varepsilon, \varepsilon, c)) = n(\varepsilon c) = c = n(\varepsilon) + n(\varepsilon) + c$.

In the $P(xa, \varepsilon)$ case, we first assume $P(x, \varepsilon)$. We have

$$\begin{aligned}
n(add(xa, \varepsilon, c)) &= n(add(x, \varepsilon, q))r && \text{with } a + c = qb + r \text{ as in the definition of } add \\
&= bn(add(x, \varepsilon, q) + r && \text{definition of } n \\
&= bn(x) + bn(\varepsilon) + qb + r && \text{by } P(x, \varepsilon) \\
&= bn(x) + 0 + a + c && \text{by definition of quotient and remainder} \\
&= n(xa) + 0 + c && \text{by definition of } n \\
&= n(xa) + n(\varepsilon) + c && \text{by defintion of } n
\end{aligned}$$

The $P(\varepsilon, xa)$ case is identical.

Now, to see $P(xa, yd)$, we first assume $P(x, y)$, $P(xa, y)$ and $P(x, yd)$. We have

$$\begin{aligned}
n(add(xa, yd, c)) &= n(add(x, y, quot(a + d + c, b))rem(a + d + c, b)) && \text{definition of } add \\
&= bn(add(x, y, quot(a + d + c, b))) + rem(a + d + c, b) && \text{definition of } n \\
&= bn(x) + bn(y) + b\ quot(a + d + c, b) + rem(a + d + c, b) && \text{by } P(x, y) \\
&= bn(x) + bn(y) + a + d + c && \text{definition of quotient and remainder} \\
&= n(xa) + n(yd) + c && \text{definition of } n
\end{aligned}$$

as required.