1. True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.

(a) \{\emptyset\} = \emptyset

(b) Every set is a subset of its power set

(c) A proof that starts “Choose an arbitrary $y \in \mathbb{N}$, and let $x = y^2$” is likely to be a proof that $\forall y \in \mathbb{N}, \forall x \in \mathbb{N}, \ldots$.

(d) The logical negation of “everybody can fool Mike” is “nobody can fool Mike”.

(e) The relation $\leq$ is an equivalence relation.

(f) If there is a bijection from $\mathbb{Q}$ to $X$ then $X$ is countable.

(g) Recall that $[X \rightarrow Y]$ denotes the set of functions with domain $X$ and codomain $Y$. Let $f : 2^S \rightarrow [S \rightarrow \{0, 1\}]$ be given by $f(X) := h$ where $h : S \rightarrow \{0, 1\}$ is given by $h(s) := 0$. $f$ is injective.

(h) $f$ as just defined is surjective.

(i) If a function has a right inverse, then the right inverse is unique.

2. Briefly and clearly identify the errors in each of the following proofs:

(a) **Proof that 1 is the largest natural number:** Let $n$ be the largest natural number. Then $n^2$, being a natural number, is less than or equal to $n$. Therefore $n^2 - n = n(n - 1) \leq 0$. Hence $0 \leq n \leq 1$. Therefore $n = 1$.

(b) **Proof that 2 = 1:** Let $a = b$.

\[ a^2 = ab \]
\[ a^2 - b^2 = ab - b^2 \]
\[ (a + b)(a - b) = b(a - b) \]
\[ a + b = b \]

Setting $a = b = 1$, we get $2 = 1$.

(c) **Proof that $(a + b)(a - b) = a^2 - b^2$:**

To prove: $(a + b)(a - b) = a^2 - b^2$

\[ a^2 - ab + ab - b^2 = a^2 - b^2 \]
\[ a^2 - b^2 = a^2 - b^2 \]

...which is true, hence the result is proved.

3. Which of these is the correct negation of $\exists x, \forall y, \exists z, \neg F(x, y, z)$?

(a) $\exists x, \exists y, \exists z, F(x, y, z)$

(b) $\exists x, \exists y, \exists z, \neg F(x, y, z)$

(c) $\forall x, \forall y, \forall z, F(x, y, z)$

(d) $\forall x, \forall y, \forall z, \neg F(x, y, z)$
4. Complete the following diagonalization proof:

Claim: \( X = [\mathbb{N} \to \mathbb{N}] \) is uncountable.

Proof: We prove this claim by contradiction. Assume that \( X \) is countable. Then there exists a function \( F : \text{FILL IN} \) that is \( \text{FILL IN} \).

Write \( f_0 = F(0), f_1 = F(1), \) and so on. We can write the elements of \( X \) in a table:

\[
\begin{array}{cccc}
 & 0 & 1 & 2 & \cdots \\
 f_0 & f_0(0) & f_0(1) & f_0(2) & \cdots \\
f_1 & f_1(0) & f_1(1) & f_1(2) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

Let \( f_D : \text{FILL IN} \) be given by \( f_D : x \mapsto \text{FILL IN} \)

Then \( \text{FILL IN} \)

This is a contradiction because \( \text{FILL IN} \).

5. Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.

(a) The set \( \Sigma^* \) containing all finite length strings of 0’s and 1’s.
(b) The set \( 2^\mathbb{N} \) containing all sets of natural numbers.
(c) The set \( \mathbb{N} \times \mathbb{N} \) containing all pairs of natural numbers.
(d) The set \( [\mathbb{N} \to \{0, 1\}] \) containing all functions from \( \mathbb{N} \) to \( \{0, 1\} \).

Be sure to include enough detail:

- If listing elements, be sure to clearly state how you are listing them;
- If diagonalizing, be sure it is clear what your diagonal construction is;
- If providing a function, make sure it is clear what the output is on a given input.

6. For any function \( f : A \to B \) and a set \( C \subseteq A \), define \( f(C) = \{ f(x) \mid x \in C \} \). That is, \( f(C) \) is the set of images of elements of \( C \). Prove that if \( f \) is injective, then \( f(C_1 \cap C_2) = f(C_1) \cap f(C_2) \) for all \( C_1, C_2 \subseteq A \).

(Hint: one way to prove this is from the definition of set equality: \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \).)

7. (a) Write the definition of “\( f : A \to B \) is injective” using formal notation (\( \forall, \exists, \text{“and”}, \text{“or”}, \text{“if \ldots then \ldots”}, =, \neq, \ldots \)).
(b) Similarly, write down the definition of “\( f : A \to B \) is surjective”.
(c) Write down the definition of “\( A \) is countable”. You may write “\( f \) is surjective” or “\( f \) is injective” in your expression.

8. Recall that the composition of two functions \( f : B \to C \) and \( g : A \to B \) is the function \( f \circ g : A \to C \) defined as \( (f \circ g)(x) = f(g(x)) \). Prove that if \( f \) and \( g \) are both injective, then \( f \circ g \) is injective.

9. For each of the following functions, indicate whether the function \( f \) is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.

(a) \( f : \mathbb{N} \to \mathbb{N} \) given by \( f : x \to x^2 \)
(b) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f : x \to x^2 \)
(c) \( f : X \to [Y \to X] \) given by \( f(x) := h_x \) where \( h_x : Y \to X \) is given by \( h_x(y) := x \).
10. [6 points] Recall that $[X \to Y]$ denotes the set of functions with domain $X$ and codomain $Y$. Let $X$ and $Y$ be nonempty sets, and let $F : [X \to Y] \to [X \to (Y \times Y)]$ be given by $F(f) := h_f$, where $h_f : X \to (Y \times Y)$ is given by $h_f(x) := (f(x), f(x))$ for all $x$.

(a) Show that $F$ is injective. Note: $g_1 = g_2$ if and only if, for all $x$, $g_1(x) = g_2(x)$.

(b) Show that $F$ is not necessarily bijective.

11. Prove that $7^m - 1$ is divisible by 6 for all positive integers $m$.

12. [6 points] Pascal’s triangle is a sequence of rows, where each entry is formed by adding the two adjacent entries from the previous row:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\ldots
\end{array}
\]

If we let $P_{n,k}$ stand for the $k$th entry in the $n$th row of Pascal’s triangle, then $P_{n,k}$ is given by the formulas $P_{1,1} := 1$, $P_{n,0} := 0$ for all $n$, and $P_{n+1,k} := P_{n,k-1} + P_{n,k}$ if $n \geq 1$.

Prove by induction on $n$ that for all $n \geq 1$, for all $k$ with $1 \leq k \leq n$, $P_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Note: The definition of $n!$ is $0! := 1$ and $n! := n \cdot (n-1)!$ for all $n \geq 1$.

13. Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

14. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:

\[
\begin{align*}
F_0 &= 1 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2
\end{align*}
\]

That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers $n$ (including 0):

\[
\sum_{i=0}^{n} F_i = F_{n+2} - 1
\]

15. Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.

16. Chapter 5 of MCS has a bunch of good induction exercises (and you can find even more by searching)

17. Find the Bézout coefficients of 5 and 12. Show your work.