Lecture 9: Countability

Last time: $|A|$ informally: the "size" or "cardinality" of $A$.

- **Defn:** $|A| \leq |B|$ means there is an injection $f : A \to B$
- **Defn:** $|A| \geq |B|$ means there is a surjection $f : A \to B$
- **Defn:** $|A| = |B|$ means there is a bijection $f : A \to B$

- **Facts:** $|A| \leq |A|, |A| = |A|, |A| \geq |A|$
- **Facts:** if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$, similarly for $\geq$ and $=$
- **Facts:** $|A| \leq |B|$ if and only if $|B| \geq |A|$
- **Facts:** (Cantor-Schroeder-Bernstein) if $|A| \leq |B|$ and $|A| \geq |B|$ then $|A| = |B|$

Recall $\mathbb{N} := \{0, 1, 2, \ldots \}$. Let $\mathbb{N}^+ := \{1, 2, \ldots \}$ (0 is not included).

- **Question:** Is $|\mathbb{N}^+| \geq |\mathbb{N}|$?
- **Question:** Is $|\mathbb{N}^+| \leq |\mathbb{N}|$?
Question: How are \(|\mathbb{N}^+|\) and \(|\mathbb{N}|\) related?

- Defn: \(|A| \leq |B|\) means there is an injection \(f: A \rightarrow B\)
- Defn: \(\mathbb{N} := \{0, 1, 2, \ldots\}\) (natural numbers, 0 is included).
- Defn: \(\mathbb{N}^+ := \{1, 2, \ldots\}\) (positive natural numbers, 0 is not included).

Question: Is \(|\mathbb{N}^+| \geq |\mathbb{N}|\) and \(|\mathbb{N}^+| \leq |\mathbb{N}|\)?

- A. \(|\mathbb{N}^+| \geq |\mathbb{N}|\) and \(|\mathbb{N}^+| \leq |\mathbb{N}|\)
- B. \(|\mathbb{N}^+| \geq |\mathbb{N}|\) but \(|\mathbb{N}^+| \not\leq |\mathbb{N}|\)
- C. \(|\mathbb{N}^+| \not\geq |\mathbb{N}|\) but \(|\mathbb{N}^+| \leq |\mathbb{N}|\)
- D. \(|\mathbb{N}^+| \not\geq |\mathbb{N}|\) and \(|\mathbb{N}^+| \not\leq |\mathbb{N}|\)
- E. Unsure

\[ f: \mathbb{N} \rightarrow \mathbb{N}^+ \]
\[ f(n) = n + 1 \]

\[ g: \mathbb{N}^+ \rightarrow \mathbb{N} \]
\[ g(n) = n - 1 \]

A set \(X\) is countable if \(|X| \leq |\mathbb{N}|\), equivalently if \(|\mathbb{N}| \geq |X|\).

\(\mathbb{N}^+\) is countable.

\[ f: \mathbb{N} \rightarrow X \]
\[ X = \{x_0, x_1, x_2, x_3, \ldots\} \]
\[ f(0), f(1), f(2), \ldots \]
Every element of \(X\) is in \(f\).
Question: is $\mathbb{Z}$ countable?

- Defn: $A$ is countable if $|\mathbb{N}| \geq |A|$ (equiv. if $|A| \leq |\mathbb{N}|$).
- Defn: the set of integers is $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots \}$

Question: Is $\mathbb{Z}$ countable?

A. Yes
B. No
C. Unsure
Question: is $\mathbb{N} \times \mathbb{N}$ countable?

A: yes $\approx 60$
B: no $\approx 37$

$f(n)$ would be defined by following this pattern for $n$ steps.

Clearly, eventually hit any given pair. So $f$ is a surjection.

\[ |\mathbb{N}| \geq |\mathbb{N} \times \mathbb{N}| \]

\[ f(0) := (0,0) \]
\[ f(4) := (2,0) \]
\[ f(8) := (3,1) \]

[Note: different suff from in wiki, but just as good]
Question: is $2^\mathbb{N}$ countable?

A: yes

B: no

C: unsure

$\exists f: \mathbb{N} \rightarrow 2^\mathbb{N}$

Claim: $|\mathbb{N}| \neq |2^\mathbb{N}|$ (i.e., $2^\mathbb{N}$ is uncountable).

Proof: by contradiction. Assume $2^\mathbb{N}$ is countable, i.e.

$\exists f: \mathbb{N} \rightarrow 2^\mathbb{N}$ surjective.

Example, if might look like:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
<th>$0 \in f(n)$?</th>
<th>$1 \in f(n)$?</th>
<th>$2 \in f(n)$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>

Surj: every set here

Create a new set $S_D$ by changing the diagonal.

$S_D = \{0,1,2,3,\ldots\}$

$S_D$ not in table, but every set in table!

contradiction.

more Wednesday