Specifying probability measures

**Defn:** A probability measure \( P_r \) on \( S \) is a function \( P_r : 2^S \to \mathbb{R} \) satisfying

Kolmogorov's axioms:

1. for all \( E \subseteq S \), \( P_r(E) \geq 0 \)
2. \( P_r(S) = 1 \)
3. if \( E_1 \cap E_2 = \emptyset \) then \( P_r(E_1 \cup E_2) = P_r(E_1) + P_r(E_2) \)

**Defn:** A probability space is a pair \((S, P_r)\) where \( P_r \) is a probability measure on \( S \).

How to give a probability space?

Flip can

\[
S := \{ \emptyset, a, b, c \}
\]

\[
P_r(\emptyset) := 0
\]
\[
P_r(\{a\}) := \rho
\]
\[
P_r(\{b\}) := \nu
\]
\[
P_r(\{c\}) := 1 - \rho - \nu
\]
\[
P_r(\{a, b\}) := \rho
\]
\[
P_r(\{a, c\}) := \frac{1 - \rho - \nu}{2}
\]
\[
P_r(\{b, c\}) := \frac{\rho \nu}{2}
\]
\[
P_r(S) := 1
\]

Simple event is an event with only one outcome.

Gives \( P_r(\emptyset) \) for all \( s \in S \), is enough to give entire \( P_r \).
Independence

**Question:** Suppose there are two elections. Suppose candidate A wins the first election with probability $1/2$ and candidate B wins the second election with probability $1/2$. What is the probability that candidate A and candidate B both win?

Let $S = \{(a, a), (a, b), (b, a), (b, b)\}$. 
Conditional probability

\[
\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)}
\]

**Definition:** \( A, B \) are independent if

\[
\Pr(AB) = \Pr(A) \cdot \Pr(B)
\]
**Defn:** \( \Pr(A \mid B) := \frac{\Pr(A \cap B)}{\Pr(B)} \)

**Question:** Suppose it rains 25% of the time. If it’s raining, I bring an umbrella 50% of the time. If it isn’t raining, I bring an umbrella 10% of the time. If you see me carrying an umbrella, how likely is it to be raining?

Let \( S = \{(r, u), (r, w), (\bar{r}, u), (\bar{r}, w)\} \)

Let \( R \) be the event “it is raining”; \( R := \{(r, u), (r, w)\} \)

\[ U := \{(r, u) , (\bar{r}, w)\} \]

Let \( U := \{(r, u) , (\bar{r}, w)\} \)

\[ \Pr(R) = \frac{1}{4} \]

\[ \Pr(U \mid R) = \frac{1}{2} \]

\[ \Pr(U \mid \bar{R}) = \frac{1}{10} \]

\[ \Pr(R \mid U) = ? \]

\[ \Pr(U) = \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{10} \]

\[ U = (U \cap R) \cup (U \cap \bar{R}) \]

\[ \Pr(R \cup U) = \frac{1}{4} \cdot \frac{1}{2} \]
Bayes’ identity

Defn: \( Pr(A \mid B) := \frac{Pr(A \cap B)}{Pr(B)} \)

Claim: \( Pr(B \mid A) = \frac{Pr(A \mid B)Pr(B)}{Pr(A)} \)

Proof:

\[
Pr(B \mid A) = \frac{Pr(A \cap B)}{Pr(A)}
\]

\[
Pr(A \mid B) = \frac{Pr(B \cap A)}{Pr(B)}
\]

\[
Pr(A)Pr(B \mid A) = Pr(A \cap B) = Pr(A \mid B) \cdot Pr(B)
\]

\[
Pr(B \mid A) = \frac{Pr(A \mid B)Pr(B)}{Pr(A)}
\]
Law of total probability

- **Defn:** \( P(A \mid B) := \frac{P(A \cap B)}{P(B)} \)

**Claim:** If the events \((A_i)\) partition \(S\), then \( P(B) = \sum_i P(B \mid A_i) P(A_i) \)

**Example:** Suppose it rains 25% of the time. If it’s raining, I bring an umbrella 50% of the time. If it isn’t raining, I bring an umbrella 10% of the time. How often do I bring an umbrella?

**Proof of claim:**

\[
Pr(B) = \sum_{i=1}^{m} Pr(B \mid A_i) \cdot Pr(A_i)
\]

the sets \( B \cap A_i \) are disjoint because \( A_i \) are disjoint.

they cover \( B \), i.e. \( B = \bigcup_i B \cap A_i \)

apply axiom 3.
Probability trees

- **Defn:** \( \Pr(A \mid B) := \frac{\Pr(A \cap B)}{\Pr(B)} \)

**Question:** Suppose it rains 25% of the time. If it's raining, I bring an umbrella 50% of the time. If it isn't raining, I bring an umbrella 10% of the time. If you see me carrying an umbrella, how likely is it to be raining?

Let \( S = \{(r, u), (r, u), (r, u), (r, u)\} \)

Let \( R \) be the event "it is raining"; \( R := \{(r, u), (r, u)\} \)

\( \Pr(R) = \frac{1}{4} \)

\( \Pr(U \mid R) = \frac{1}{10} \)

\( \Pr(U \mid \bar{R}) = \frac{1}{2} \)

Let \( U \) be the event "I am carrying an umbrella".

\[ U := \{(r, u), (r, u), (r, u), (r, u)\} \]

**Bayes' identity:**

\[ \Pr(U \mid R) \cdot \Pr(R) = \frac{\Pr(U \cap R)}{\Pr(U)} \]

**Law of total prob.**

\[ \Pr(U) = \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{10} \]

\[ U = (U \cap R) \cup (U \cap \bar{R}) \]

\[ \Pr(RNU) = \frac{1}{4} \cdot \frac{1}{2} \]