Lecture 7: Injectivity and Inverses

- Properties of functions
  - Injectivity, Surjectivity, Bijectivity

- Relationship to inverse functions

Applications:

- Database systems: view-update problem
  
  Database 1 \rightarrow View
  
  Database 2

  Query

  User wants to change view, can we update DBs?

- Do file format conversions lose data?

- How to implement "undo"?
injective

Not injective:
there are inputs
that give same output.

onto

Not 1-1:
there is some
output that
doesn't get "hit".

\[ a \rightarrow 1 \]
\[ b \rightarrow 2 \]
\[ c \rightarrow 3 \]
\[ d \rightarrow 4 \]

is \( \neq \) \( \Rightarrow \)
not \( \neg \Rightarrow \)
different

Defn: \( f: A \rightarrow B \) \( \triangleright \) injective \( \iff \) no two inputs
give same output.

\[ \forall x_1, x_2 \in A, \ f(x_1) \neq f(x_2) \] \( \chi \)
Not quote: \( f(a) = f(a) \).

\[ \forall x_1, x_2 \in A, \text{ if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2) \] \( \checkmark \)

Contrapositive of "if \( P \) then \( Q \)"
is "if \( \neg Q \) then \( \neg P \)"
Contrapos. always equivalent to orig.

\[ \forall x_1, x_2 \in A, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2 \]
\[ \begin{align*}
& a \rightarrow 1 \\
& b \rightarrow 2 \\
& c \rightarrow 2 \\
& \text{is surjective} \\
& \text{i.e., every output gets hit} \\
& \text{Def: } F: A \rightarrow B \text{ is surjective if} \\
& \forall y \in B \exists x \in A \text{ such that } f(x) = y.
\end{align*} \]
Claim: if $f$ is injective, then $f$ has a left inverse.

Claim: and vice-versa (i.e. if $f$ has a left inverse, then $f$ is injective).

Claim: $f$ is surjective if and only if $f$ has a right inverse.

Claim: $f$ is bijective if and only if $f$ has a two-sided inverse.

We'll prove some in lecture, some on HW; you may use them (unless asked to prove them, of course!)
Claim: if $f \circ f = \text{id}$, then $f$ has a LI.

Proof: Choose an arb. $F : f : A \to B$, assume $f$ is injective. We wish to show that there exists a function $g : B \to A$ with $g \circ f = \text{id}$.

Let $g : B \to A$ be given as follows:

- On input $y$, if $\exists x \in A$ with $f(x) = y$, let $g(y) = x$.
- Otherwise, choose any $x_0 \in A$, let $g(y) = x_0$.

Need to check that $g \circ f = \text{id}$,

i.e. wish $\forall x \in A$, $g(f(x)) = x$.

Choose arb. $x \in A$.

Since $\exists x \in A$ with $f(x) = f(x')$, we're using the first case in define of $g$.

... then $g(f(x)) = f(x) = x$.

Need to check that this $x = x'$ equals this $x' = x$.

Know $f(x) = f(x')$, so since $f$ is injective, $x = x'$.

So $g(f(x)) = x' = x$. 
To prove if $f$ is surj/bij then $f$ has right /2-sided inverse. Draw best inverse possible, check that it is right /2-sided inverse.

To prove if $f$ has left/right/2-sided inv. then $f$ is inj/surj/bij:

- Follow def's & proof techniques.
  e.g. def of injective is a "for all", so choose arb. $x, x_1$. You then need to prove "if...then" so assume if, prove then, etc.