Lecture 2: Sets

- Definitions and Notation
- Examples of sets
- Relationships between sets
- Operations on sets
Sets

Def: A set is a collection of things.

Every thing $x$ is either in any set $A$ or not in $A$.

Notation: we write $x \in A$ to mean $x$ is in $A$, $x \notin A$ means $x$ is not in $A$.

If $x \in A$, we say that $x$ is an element of $A$.

Ex: The set containing Prof. George and his pencil.

Ex: Let $A = \{1, 2, 3\}$

"the set containing 1, 2, 3"

:= means "is defined as"

$x \in A$ if $x = 1$ or $x = 2$ or $x = 3$

$1 \in A$, $2 \in A$, $3 \in A$. Prof. George $\notin A$.

Ex: Let $N = \{0, 1, 2, \ldots\}$

the natural numbers.

$x \in N$ if $x = 0$ or $x = 1$ or $x = 2$ or $\ldots$

Ex: Let $\emptyset = \{\}$ denotes the empty set.

for any $x$, $x \notin \emptyset$. 
Set comprehension

Notation: \( \{ x \mid P \} \) means
"the set of all \( x \) 'satisfying' \( P \)" "such that"

\[ y \in \{ x \mid P \} \text{ if } P \text{ is true when } y \text{ is substituted for } x. \]

How to write 'the set of all people in this room'.

\[ \text{Ex: let People:= } \{ x \mid x \text{ is a person and } P \text{ is in this room}. \] \]

\[ y \in \text{People if } y \text{ is a person and } y \text{ is in the room}. \]

write \[ \{ 1, 2, 3 \} \] using set comprehension notation?

\[ \{ 1, 2, 3 \} = \{ x \mid x=1 \text{ or } x=2 \text{ or } x=3 \} \]
Set equality

**Def.** If $A$ and $B$ are sets, we say $A = B$ if whenever $x \in A$, $x \in B$, and also, whenever $x \in B$, $x$ is also in $A$.

**Question:** Does order of elements in a set matter?

$\{1, 2, 3\} \ ? \ \{3, 1, 2\}$

Yes, these are equal.

If $x = \{1, 2, 3\}$, then either $x = 1$ or $x = 2$ or $x = 3$.

In each case, $x \in \{3, 1, 2\}$.

Similarly, if $x \in \{3, 1, 2\}$, then either $x = 1$ or $x = 2$ or $x = 3$.

In each case, $x \in \{1, 2, 3\}$.

Thus $\{1, 2, 3\} = \{3, 1, 2\}$.

**Question:** Do duplicates matter?

$\{1, 1, 2, 3\} \ = \ \{1, 2, 3\}$

Proof: Similar to above.

So duplicates don't matter.

In general, order is irrelevant.
Venn diagrams:

- Representing sets, as all points in the circled region.

Caution: Not every set can be drawn in a Venn diagram.

- Is \( B \) inside \( A \)?
- Is \( B \) a subset of \( A \)?

Definition: We say \( A \) is a subset of \( B \)
- if for any \( x \in A \), \( x \) is also in \( B \).
  (Written \( A \subseteq B \))

Note: \( A \subseteq A \).

Definition: \( A \) is a proper subset of \( B \)
- if \( A = B \) and \( A \neq B \).
  Write \( A \subset B \)

Note: \( c \) is defined differently in different sources, avoid.

In this example, \( B \subseteq A \) but \( A \not\subseteq B \)
**Power set**

**Def:** If $A$ is a set, the power set of $A$ (written $2^A$) is the set of all subsets of $A$.

i.e. $2^A := \{ B \mid B \subseteq A \}$

**Ex:** $2^{\{1,2,3\}} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

$\emptyset \in 2^{\{1,2,3\}}$

$\emptyset \neq 3\emptyset^3$

\[ \emptyset \neq 3\emptyset^3 \]

**Ex:** $\emptyset \in 2^{\{1,2,3\}} \Rightarrow \emptyset \subseteq 2^{\{1,2,3\}}$, in fact for any $A$, $\emptyset \subseteq A$. 

"be careful not to write things like $\emptyset = 3\emptyset^3$"
Question: is $\emptyset \subseteq \emptyset$?

to check: look at all elts of $\emptyset$,
if each is in $\emptyset$, answer is yes,
if any elt of $\emptyset$ is not in $\emptyset$,
answer is no.

$\emptyset \in \emptyset$? No, by defn of $\emptyset$: for all $x$, $x \notin \emptyset$.

Answer: $\emptyset \subseteq \emptyset$, because there are no elements in LHS that are not in RHS.