Lecture 29: Combinatorics

- How to compute & elts of a set without enumerating them

- Def: |N|, Finite

- Sum & product rule, tree diagrams

Applications:

- Useful for evaluating running time of algorithms
- Relationships between &s of things gives bijections, can lead to surprising problem transformations.

  e.g. binary $\rightarrow$ unary data structure design

  e.g. paths on spheres, tori, etc $\rightarrow$ type equivalences
Definition: If $A$ is a set and $n \in \mathbb{N}$, we say $|A| = n$, if $A$ has $n$ elements, i.e.

\[ |A| = |\{1, 2, 3, \ldots, n^3\}|, \text{ i.e. if a bij. } f : A \rightarrow \{1, 2, 3\} \]

In this case, we say $A$ is finite.

Example: $A = \{a_1, a_2, \ldots, a_5\}$

Claim: If $|A| = m$ and $|A| = n$, then $m = n$.

\[ |\{1, 2, 3\}| = |\{1, 2, 3\}| \]

If $m < n$, find $i \neq j$ with $f(i) = f(j)$, so $f$ is not injective, (contradiction).

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**Claim (sum rule):** If $A \cap B$ are disjoint, then

\[ |A \cup B| = |A| + |B| \text{ (Note: implicitly assume that } A, B, \ldots \text{ are finite)} \]

Assume $|A| = n$, $|B| = m$

Let $h(i) := f(i)$ if $1 \leq i \leq n$

Let $g(i) := g(n+i)$ if $i > n$. Claim: $h$ is a bijection.
Claim: \(|A \times B| = |A| \cdot |B|\)

\(\text{PF 1:}\) we could construct a bijection

\(f: \{1, 2, \ldots, n \cdot m\} \rightarrow A \times B\)

\[\begin{align*}
\text{row 1:} & \quad (a_1, b_1) \quad (a_1, b_2) \quad (a_1, b_3) \quad \ldots \quad (a_1, b_m) \\
\text{row 2:} & \quad (a_2, b_1) \\
\vdots & \\
\text{row } n: & \quad (a_n, b_1) \\
\end{align*}\]

\(n+1 \rightarrow (a_2, b_1)\).

\(\text{rule: } f(i) = (a_{\text{round}(i-1,n)} + 1, b_{\text{rem}(i-1,n)+1})\)

\(\text{PF 2:}\) to construct an element of \(A \times B\),

1. choose an elt \(a\) of \(A\) - \(n\) options
   - choose \(a\)

2. choose an elt \(b\) of \(B\) - \(m\) options
   - choose \(b\)

3. output \((a, b)\)

\(\text{Knew } A \times B = \{(a, b) \mid b \in B\} \leq m\) elts

\[\text{U} \left\{(a, b) \mid b \in B\right\} \leq m\) elts \]

\(\text{disjoint}.

\[\text{U} \left\{(a, b) \mid b \in B\right\} \leq m\) elts.

So \(|A \times B| = |B| + |B| + \ldots + |B| = n \cdot m\).

\(\text{by sum rule}\)
Sum rule in terms of processes:

1. make a choice, 2. based on that choice construct either a or b possibilities,

3. output contains either.

Product rule:

1. make a choice, 2. make another choice with l options, 3. output both choices,

total # options: a·b.
Example 1: How many possible license plates exist if a l.p. has 3 letters followed by 4 digits?

Uppercase

1. Choose 1st letter
2. Choose 2nd letter
3. Choose 3rd letter
4. Choose 1st digit

26 options, 26 options, 26 options, 10 options, 10 options, 10 options, 10 options.

Total: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4$