Lecture 31: Probability

- Defn's
- Examples

Defn A **sample space** S is a set.

An **outcome** is an element of S.

An **event** is a subset of S.

Ex: Rolling a six-sided die:

$S = \{1, 2, 3 \ldots, 6\}$,

outcomes we're willing to consider
(could actually be impossible)
like Codomain are "possible" outputs.

$E = \{2, 4, 6\}$ represent "I roll an even #".

Def$^{\underline{n}}$: A probability measure Pr is a function

$$Pr : 2^S \to \mathbb{R}, \quad \text{satisfying}$$

Pr inputs are events

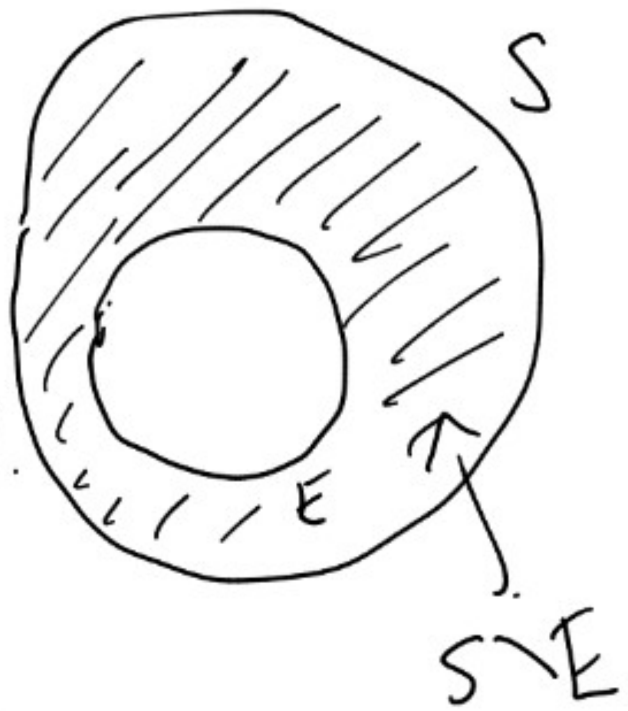
1. $\forall E, \; Pr(E) \geq 0$.
2. $Pr(S) = 1$.
3. if $E_1 \cap E_2 = \emptyset$, then

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2).$$

Kolmogorov's axioms.

<u>Claim:</u> $\forall E,\ \Pr(E) \leq 1$.

<u>Pf:</u> Know $E \cap (S \setminus E) = \emptyset$.



So $\Pr(\underbrace{E \cup (S \setminus E)}) = \Pr(E) + \Pr(S \setminus E)$.
(by ax. 3).

LHS $= \Pr(S) = 1$. by rule 2.

Also $\Pr(S \setminus E) \geq 0$ by rule 1

$1 = \Pr(E) + \Pr(S \setminus E) \geq \Pr(E)$. ✓

by above.

adding $\Pr(E)$ to

<u>In fact</u>:

$$1 = \Pr(E) + \Pr(S \setminus E)$$

So $$\Pr(E) = 1 - \Pr(S \setminus E).$$

Defⁿ A probability space is a pair (S, Pr) where S is a sample space, Pr is a prob. measure on S.

Ex: one possible way to represent rolling a die

$S = \{1, 2, \dots, 6\}$

$Pr(\{1\}) := 1/6$, $Pr(\{2\}) = 1/6$ ... $Pr(\{6\}) := 1/6$.

Note: For finite sets, can infer $P(E)$ by writing $E = \{a_1\} \cup \{a_2\} \cup \dots \cup \{a_n\}$, use axiom (3) to get $Pr(E)$ from $Pr(\{a_i\})$.

Note: In this case, $Pr(E) = Pr(\{a_1\}) + \dots + Pr(\{a_n\})$
$= |E| / |S|$.

Warning!! this is not the only Prob. Measure! Don't assume all probs. are same (i.e. $Pr(\{a\}) = Pr(\{b\})$)

Q: Model rolling 2 dice, what is S?

A: $S = \{(1,1) \ldots (1,6), (2,1) \ldots (2,6) \cdots (6,6)\}$

$= \{1, \ldots, 6\} \times \{1, \ldots, 6\}$

A2: $S = \{2, 3, \ldots, 12\}$ (possible sums)

but inconvenient. Pr is complicated with this setup.

both work, but hard to define $\Pr(\cdots)$ for 2^{nd} option

Q: Experiment: select someone in room (everyone w/ equal prob), measure height.

S = ?

{height of shortest, height of next ... , height of tallest}.

S = ? $\{(\overset{n}{\text{name}}, \overset{h}{\text{height}}) \mid$ height of person with name $n = h\}$.

S = {people in the room}.

all work, but hard to define Pr for first two.