Lecture 9: Countability

- Defn: Countable, uncountable
- Rational #s ($\mathbb{Q}$) is countable
- Real #s ($\mathbb{R}$) is uncountable (there are more reals than rationals)

Applications

Why is "digital" so important?
- What does "digital" mean, anyway?
**Def.** A set $X$ is **countable** if $|X| \leq |\mathbb{N}|$ (equivalently, if $|\mathbb{N}| \geq |X|$)

\[ X = \{ x_0, x_1, x_2, \ldots \} \]

$f : \mathbb{N} \to X$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_0$</td>
</tr>
<tr>
<td>1</td>
<td>$x_1$</td>
</tr>
<tr>
<td>2</td>
<td>$x_2$</td>
</tr>
<tr>
<td>3</td>
<td>$x_3$</td>
</tr>
<tr>
<td>4</td>
<td>$x_4$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

every elt of $X$ gets hit by this fn (surj).

$\sqrt{\text{\textbf{N is countable}}} \quad \sqrt{\text{\textbf{N u \{-1\} is ctbl}}} \quad \sqrt{\text{\textbf{Z is ctbl}}} \quad \sqrt{\text{\textbf{Q (the set of rational is \{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\} ) is ctbl}}} \quad \sqrt{\text{\textbf{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots is ctbl}}}$
Claim: \( \mathbb{Q} \) is countable

Claim*: \( \mathbb{Q}^+ \) is countable, where \( \mathbb{Q}^+ = \{ x \in \mathbb{Q} \mid x > 0 \} \) is countable.

Proof: WTS \( |\mathbb{Q}| \leq |\mathbb{N}| \) or \( |\mathbb{N}| \geq |\mathbb{Q}| \),

i.e. \( \exists \) a surj \( f: \mathbb{N} \rightarrow \mathbb{Q} \)

let \( f: \mathbb{N} \rightarrow \mathbb{Q} \) be given by

\[ f(n) = \frac{n}{2} \]

not surjective! "hit" \( \frac{1}{2} \).

\[ f(n) = \begin{cases} \frac{1}{n} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \]

Also not surjective.

Idea: to compute \( f(n) \), we'll traverse the table of rationals as shown, \( n \) steps, then \( f(n) \) is the resulting fraction.

\( f \) is surjective, because every elt of \( \mathbb{Q}^+ \) is in the table at least once, every position in the table gets "hit" by \( f \).
Claim: \( \mathbb{R} \) (the set of real \( \mathbb{R} \)) is not countable.

Proof: WTS \( |\mathbb{N}| \neq |\mathbb{R}| \), i.e. \( \not\exists \) a surjection \( f: \mathbb{N} \to \mathbb{R} \). For the sake of contradiction, assume \( \exists \) a surjection \( f: \mathbb{N} \to \mathbb{R} \).

For example, \( f \) might look like:

\[
\begin{array}{c|c}
 n & f(n) \\
\hline
 0 & 0.000000 \\
 1 & 3.141592 \\
 2 & 0.399288 \\
 \vdots & \vdots \\
 i & \vdots \\
\end{array}
\]

Let \( \chi_0 \) be the \( \text{symbolic} \) of \( \chi \), then \( \chi \neq f(\chi) \) why?

To get \( \chi \)th digit of \( \chi \), add 5 (wrap around) to i\text{th} digit of \( f(i) \).

I claim: \( \chi \) cannot be \( f(k) \) for any \( k \), because \( \chi \) and \( f(k) \) differ in the \( k \)th digit. So

\[
|\chi - f(k)| > 0.000000 \ldots k\text{ zeros.}
\]

This contradicts the fact that \( f \) is surjective, so there cannot be a surj. \( f: \mathbb{N} \to \mathbb{R} \), so \( |\mathbb{N}| \neq |\mathbb{R}| \).