Lecture 8: Cardinality (starting with a proof or two about inverses)

**Defn:** \( g : B \to A \) is a left inverse if \( f : A \to B \) if \( g \circ f = \text{id}_A \)

**Defn:** \( f : A \to B \) is injective if for all \( x_1, x_2 \in A \), if \( f(x_1) = f(x_2) \) then \( x_1 = x_2 \)

**Claim:** If \( f \) has a left inverse than \( f \) is injective.

**Proof:** Choose arb. sets \( A, B, \) and \( f : A \to B \).

- **(a)** Assume \( f \) has a left inverse, i.e., \( A \) has a left inverse \( \text{id} : \).
- **(b)** Let's call it \( g : B \to A \), with \( g \circ f = \text{id} \) on \( A \).
- **(c)** \( g \) is injective, i.e., wts \( g(f(x_1)) = g(f(x_2)) \) then \( x_1 = x_2 \).

**Proof:** Choose arb. \( x_1, x_2 \in A \).

Well since \( g \circ f = \text{id} \), and \( f(x_1) = f(x_2) \), we have \( x_1 = g(f(x_1)) = g(f(x_2)) = x_2 \).

**A.** prove a “for all”

choose an arbitrary …

**B.** use a “for all”

in particular, with \( x = \ldots \)

**C.** prove an “if then”

Assume …

**D.** use an “if then”

Since …

**E.** use a “there exists”

Call it \( x \) …
(A) I'm here... let's have lecture

(B) Class is cancelled... put up a video and stay for extra office hours
The converse is also true

**Defn:** $g : B \to A$ is a **left inverse** if $f : A \to B$ if $g \circ f = id_A$

**Defn:** $f : A \to B$ is **injective** if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$

**Claim:** If $f : A \to B$ is injective and $A \neq \emptyset$ then $f$ has a left inverse

**Proof:** Choose an arbitrary $A \neq \emptyset$, $B$, and $f : A \to B$. We want to show that $f$ has a left inverse. Let $g : B \to A$ be given as follows:

(i) - if $\exists x \in A$ with $f(x) = y$, define $g(y) = x$.

(ii) - otherwise, choose any $x_0 \in A$, let $g(y) = x_0$.

We need to check: $g : B \to A$ is well-defined.

- $g$ clearly gives an output for each input.

- Once reader chooses $x_0$, then $g$ is unambiguous.

Choose arb. $x_0 \in A$. Then $g(f(x))$ is defined using rule (ii), since $\exists x \in A$ with $f(x) = f(x)$.

Then $g(f(x)) = x$. Well, since $f(x) = f(x)$, $x' = x$, so $g(f(x)) = x' = x$, as required.
Other facts (proofs are good exercises!):

- **Claim:** $f$ is injective if and only if $f$ has a left inverse
- **Claim:** $f$ is surjective if and only if $f$ has a right inverse
- **Claim:** $f$ is bijective if and only if $f$ has a two-sided inverse
Cardinality:

Idea (NOT defn): the cardinality of $A$, (written $|A|$) is the size of $A$.

Note: Until further notice, $|A|$ will not be defined. Using counting doesn’t work with infinite sets, and we want proofs that do.

We will define $|A| \leq |B|$, $|A| \geq |B|$, and $|A| = |B|$, however

Defn: $|A| \leq |B|$ means (question from reading):

A. there exists an injection $f : A \rightarrow B$
B. there exists a surjection $f : A \rightarrow B$
C. there exists a bijection $f : A \rightarrow B$
D. $A$ has fewer elements than $B$
E. unsure/other

$|A| \leq |B|$ means \( \exists f : A \rightarrow B \)

$|A| \geq |B|$ means \( \exists f : A \rightarrow B \)

$|A| = |B|$ means \( \exists f : A \rightarrow B \)
Things to check:

**Defn:** $|A| \leq |B|$ means that there exists an injection $f : A \rightarrow B$

**Defn:** $|A| \geq |B|$ means that there exists a surjection $f : A \rightarrow B$

**Defn:** $|A| = |B|$ means that there exists a bijection $f : A \rightarrow B$

**Claims:**

- $|A| = |A|$
- if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$
- if $|A| \leq |B|$ then $|B| \geq |A|$ (and vice versa)
- if $|A| \leq |B|$ and $|A| \geq |B|$ then $|A| = |B|$ (○)
- ... others

\[ \text{need proofs} \]