Lecture 7: ’jectivity and inverses

Announcements:

- Reminder: HW1 due Friday; today is last day for HW1 office hours
- HW2 out now, due a week from Friday

Today:

- Definition of function equality
- Operations for combining functions
- Relationship between ’jectivity and inverses
Function equality

**Question:** What does it mean for two functions to be equal?

**Definition:** Two functions $f : A \rightarrow B$ and $g : A \rightarrow B$ are equal (written $f = g$) if ...

For every $x \in A$, $f(x) = g(x)$.

For all elements of $B$.
The idea behind inverses

We can “flip over” the following function to get another function:

\[ f : A \rightarrow B \]

\[ a \rightarrow 1 \]
\[ b \rightarrow 2 \]
\[ c \rightarrow 3 \]

\[ g : B \rightarrow A \]

\[ 1 \rightarrow a \]
\[ 2 \rightarrow b \]
\[ 3 \rightarrow c \]

\[ g(f(a)) = a \]
\[ g(f(b)) = b \]
\[ g(f(c)) = c \]

\[ g \text{ is a "solver" for } f \]

\[ f(x) = y \text{ with } f(x) = 1 \]

For all \( x \in A \),
\[ f(g(y)) = y \]
Defn: If \( f : B \rightarrow C \) and \( g : A \rightarrow B \) then the \textbf{composition} of \( f \) and \( g \) (or “\( f \) of \( g \)”, written \( f \circ g \)), is the function \( (f \circ g) : A \rightarrow C \) given by \((f \circ g)(x) := f(g(x))\) for all \( x \in A \).

**Question:** Let \( f = \begin{array}{ccc}
1 & 2 & 3 \\
a & b & a
\end{array} \) and \( g = \begin{array}{ccc}
1 & 2 & 3 \\
1 & a & b
\end{array} \). What is \( f \circ g \)?

- A. \( \begin{array}{ccc}
1 & 1 & 1 \\
a & a & a \\
b & b & b
\end{array} \)
- B. \( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 2 & 3
\end{array} \)
- C. \( \begin{array}{ccc}
a & 1 & 1 \\
b & 2 & 2 \\
1 & 3 & 3
\end{array} \)
- D. Something different
- E. Unsure
Different kinds of inverses

**Defn:** The identity function of a set $A$ (written $id_A$, or just $id$ if $A$ is clear from context) is the function $id_A : A \to A$ given by $id_A(x) := x$.

**Defn:** If $f \circ g = id$ then $f$ is a left inverse of $g$, and $g$ is a right inverse of $f$.

**Defn:** If $f$ is both a left- and a right-inverse of $g$ then $f$ and $g$ are two-sided inverses.

\[ \begin{align*}
&\begin{array}{ccc}
1 & \leftrightarrow & 2 \\
2 & \leftrightarrow & 3 \\
3 & \leftrightarrow & 1 \\
\end{array} \\
\text{f : A \to B} & \quad & \text{g : B \to A} \\
& \quad & \text{g \circ f = id}.
\end{align*} \]

\[ \begin{align*}
\text{(g \circ f)}(x) &= \text{id}(x) \\
\text{for all } x \in A, \\
g(f(x)) &= x.
\end{align*} \]

$g$ is a right inverse of $f$. $g$ "solves for inputs of" $f$: for all $y \in B$, $f(g(y)) = y$. If $f \circ g = id$, then $g(f(x)) = x$. 

$g$ is a left inverse of $f$. $g$ "undoes" $f$: for all $x \in A$, $g(f(x)) = x$. If $g \circ f = id$, then $f \circ g = id$.
not injective (surjective)

not a function (ambiguity)

is a function

\[ f \circ g = \text{id} \]

\[ g \circ f \neq \text{id} \]

\[ g \text{ is a left inverse} \]

right!

\[ g \text{ is not a right inverse} \]

\[ \text{right!} \]

\[ g \text{ is not a right inverse} \]

\[ g \text{ is not total} \]

\[ g \text{ is a l.i.} \]
Facts:

- **Claim:** $f$ is injective if and only if $f$ has a left inverse
- **Claim:** $f$ is surjective if and only if $f$ has a right inverse
- **Claim:** $f$ is bijective if and only if $f$ has a two-sided inverse

(proofs: later)