Lecture 12: Equivalence classes.

- Definitions
- Examples
- Relationship to partitions
- Well-defined \( f^* \)s

Applications

- in 2800: "modular \#s" defined as equiv. classes

- elsewhere: "connected components" are an important property of graphs, and are equivalence classes

- objects with redefined "equals" \( f^* \) are representatives of equivalence classes; impacts storage & \( f^* \) specifications
A/R = \{ [a]_R \mid a \in A \}
= \{ [a], [b], [c], [d], [e], [f] \}
= \{ [a], [b], [c], [d], [e], [f] \}
= \{ [a], [b], [c], [d], [e], [f] \}

A = \{ a, b, c, d, e, f \}
\{ a, b, c \} \{ d, e \}
\{ d \} \{ e \}
\{ b \} \{ a, b, c \}
\{ a \} \{ a, b, c \}

Example: Think of "is-a-relative-of" relation or set of people.
- is an equivalence relation
  - Reflexive: everyone is related to themselves
  - Symmetric: if xRy then yRx
  - Transitive: relative of my relative is also my relative

families are equivalence classes.
if $A$ is a set, $R$ is an equivalence relation on $A$, then the equivalence class of $a \in A$ is the set of all $b \in A$ with $bRa$.

notation: $[a]_R$ is equiv. class of $a$

(sometimes $[a]$ if $R$ clear)

$[\text{me}] = \text{my family} = \{\text{my sister}\}$

$[a]_R := \{b \in A \mid bRa\}$

\[\text{Defn.}\ : \text{The set of all equiv. classes of } A \text{ under } R \text{ is called "} A \text{ modulo } R \text{" or "} A \text{ mod } R \text{" (written } A/R \text{)} \]
\( R = \text{family relationships} \)
\( A = \text{people} \)

Let \( f : \frac{A}{R} \rightarrow X \) be given by

\[
f([a]) := a
\]

\[
f([a]) := a's \text{ eye color}
\]

alice \( R \) bob

\[
[a] = [b]
\]

\[
f([alice]) = \text{alice's eye color = green}
\]

\[
f([bob]) = \text{bob's eye color = blue}
\]

\( f \) is ambiguous (not a \( f^* \)).

**Term:** \( a^* A \), \( a \) is a representative of \([a]\)

**Check:** When defining \( f^* : f \) from \( A/R \) to another set \( X \), check that \( f \) is unambiguous (well-defined)

important to check when you redefine "equals"
Define sets $X_1, X_2, \ldots, X_n$ partition a set $A$ if

1. $X \cup X_1 \cup X_2 \cup \ldots \cup X_n = A$ (every elt of $A$ is in some partition)
2. $X_i \cap X_j \neq \emptyset$ then $X_i = X_j$

Claim: $A/R$ partitions $A$.

Proof: Let $A = \bigcup_{i \in \alpha} [a_i] \cup [a_2] \cup \ldots$

Need $A = \bigcup_{i \in \alpha} [a_i] \cup [a]$ and $A \subseteq \text{RHS}$

Choose $x \in [a]$ if $x \in [a_i]$, for some $a_i \in A$, so $x \in [a_i] \subseteq \text{RHS}$.

Choose $X \subseteq A$, WTS $x \in \text{RHS}$, we can do this by showing $x \in [a_i]$ for some $a_i \in A$, $x \in X$, because $x \in X$ (since $R$ is reflexive).

2. WTS if $[a] \cap [b] \neq \emptyset$ then $[a] \\ [b] \\

Assume $[a] \cap [b] \neq \emptyset$.

Then $[a] \cap \text{cr}_A \cap \text{cr}_B$

Then $\text{cr}_A \subseteq [a]$ and $\text{cr}_B \subseteq [b]

WTS $[a] = [b]$, i.e. $\forall x \in [a], x \in [b]$ and $\forall x \in [b], x \in [a]$.

Choose $x \in [a]$. Then $x \in [a] \subseteq [b]$, i.e. $\forall x \in [a], x \in [b]$.

Since R is irreflexive and cr, we have $\text{cr}_A \subseteq [a]$, since $R$ is transitive, we have $\text{cr}_B \subseteq [b]$.

Since $R$ is irreflexive, we also have $\text{cr}_A \subseteq [a]$, so $x \in [b]$ by transitivity.

$[b] \subseteq [a]$.