Combinatorics

**Question 1:** A standard license plate consists of 4 letters followed by three digits. How many possible standard license plates are there?

**Question 2:** A vanity plate consists of either 4, 5, or 6 characters, each of which is either a letter or a digit. How many possible vanity plates are there?

**Question 3:** Every license plate is either a vanity plate or a standard plate. How many license plates are there?
Finite cardinality definitions

- Recall: $|A| = |B|$ means there exists a bijection $f : A \rightarrow B$

**Def**: if $A$ is a set and $n \in \mathbb{N}$, then

$$|A| = n \text{ means } |A| = \{1, 2, 3, \ldots, n\}$$

This means if $|A| = n$ then $A = \{a_1, a_2, \ldots, a_n\}$

(with $a_i \neq a_j$ unless $i = j$)

**Def**: if $|A| = n$ for some $n$, then we say $A$ is **finite**

**Note**: if $|A| = n$ and $|A| = m$ then $n = m$. 
**Sum rule**

**Claim:** If \( A \) and \( B \) are disjoint then \( |A \cup B| = |A| + |B| \)

**Question:** What does \( |A| + |B| \) mean?

**Restated claim:** if \( |A| = k \) and \( |B| = l \) then \( |A \cup B| = k + l \).

**Proof:**

\[ f \]
\[ g \]
\[ A \]
\[ B \]
\[ A \cup B \]

let \( h(n) = \begin{cases} f(n) & \text{if } n \in A \\ g(n) & \text{if } n \in B \end{cases} \)

check: \( h \) is a bijection.
Product rule

Claim: \(|A \times B| = |A| \cdot |B|\), i.e. if \(|A| = k\) and \(|B| = \ell\) then \(|A \times B| = k\ell|.

Proof 1:

\[ A = \{a_1, a_2, \ldots, a_k\} \]
\[ B = \{b_1, b_2, \ldots, b_\ell\} \]

\[
\begin{align*}
(a_1, b_1) &

(a_1, b_2) &

(a_1, b_3) &

(a_1, b_\ell) \\
(a_2, b_1) &

(a_2, b_2) &

&

(a_2, b_\ell) \\
&

(a_k, b_1) &

&

(a_k, b_\ell)
\end{align*}
\]

\[ f(1) = (a_1, b_1) \quad f(2) = (a_1, b_2) \quad \ldots \]
\[ f(\ell) = (a_k, b_\ell) \]
Product rule

Claim: \(|A \times B| = |A| \cdot |B|\), i.e. if \(|A| = k\) and \(|B| = \ell\) then \(|A \times B| = k\ell\).

Proof 2:

How to construct a pair \((a, b)\)?

\[
A \times B = \bigcup_{b \in B} \{ (a, b) : a \in A \}.
\]

Process:

1. choose \(a\) from \(A\) (\(k\) options)
2. choose \(b\) from \(B\) (\(\ell\) options)
3. output \((a, b)\)

\(|A \times B| = k + l + l + \ldots + l = k\ell\).
Counting using processes

To find $|A|$:

- Describe a process for constructing an element of $A$ (picture helps).
- Count the number of options for each choice you make.
- Use the sum rule, product rule, etc. to combine those numbers to arrive at $|A|$.

**Note:** For this section of the course (i.e. when doing combinatorics problems), we are most interested in your description of the process and the rules you use to combine things together than we are in either a proof of the existence of a bijection or computing numbers.
Suppose \(|A| = n\). Let \(A = \{a_1, a_2, a_3, \ldots, a_n\}\).

What is \(|2^A|\)?

- Process for constructing an elt. of \(2^A\)

Choose \(B \subseteq A\), \(a \in B\).

n \(\leq\) \(\sum_{k=1}^{n} m_k\)