Lecture 20: Structural Induction

- inductively defined sets
- inductively defined funs
- proofs by structural induction
- how to prove "for all strings x, P(x)" or
  "for all trees t, P(t)" or
  "for all programs p, P(p)"

Applications:
- in 2800: reasoning about strings, expressions, logic, proofs
- elsewhere: inductive defs are important in functional programming and programming language design.
BNF (Backus-Naur Form)

Notation:

```
  x ::= ε | xa
  a ∈ A
```

Strings:
- `ε` is an empty string.
- `x` is a string.
- `ab` is a string with `b` at the end.
- `w` is a string with `d` at the end.
- `a` is a string.
- `ε` is the empty string.

Trees:
- `ε` is a tree by rule 1.
- `a` is a tree by rule 1.
- `ab` is a tree by rule 2.
- `w` is a tree by rule 1.

```
0 = Z
1 = S Z
2 = S(S Z)
```

Numbers:
- 0 is the successor of -1
- 1 is the successor of 0
- 2 is the successor of 1
- 3 is the successor of 2

```
eE Expr ::= n | e1 + e2
  if e1 = 0 then e2 else e3
  n ∈ N
```

Expressions:
- `n` is a number.
- `e1 + e2` is an expression.
- `if e1 = 0 then e2 else e3` is an expression.
- `(if e1 = 0 then e2 else e3)` is also an expression.

```
if Z = 0 then H + 1 else 2) + 3
```
To define a $f_2$ from $X \rightarrow Y$, we define $f$ for each kind of value formed by different rules, defined in terms of $f$ of the substructures of $x$.

Example: $\text{len} : \Sigma^* \rightarrow \mathbb{N}$

- $\text{len}(\varepsilon) = 0$
- $\text{len}(\alpha x) = \text{len}(x) + 1$

where $\alpha$ is a substructure of $x$.

Example: the height of a tree is given by:

- $h(\varepsilon) = 0$
- $h(\alpha x) = \max(h(x_1), h(x_2)) + 1$

Example: The # of nodes in a tree $n : \text{Trees} \rightarrow \mathbb{N}$

- $n(\varepsilon) = 0$
- $n(\alpha x) = n(x_1) + n(x_2) + 1$

Claim: $\forall x \in \text{Trees}, n(x) \leq 2 \cdot h(x) - 1$

Proof: We'll prove by structural induction on $t$. Need to show $P(\varepsilon)$ and $P(\alpha x)$, assuming $P(t_1)$ and $P(t_2)$.

$P(\varepsilon)$: WTS $n(\varepsilon) \leq 2h(\varepsilon) - 1$

- Well $n(\varepsilon) = 0$ by def.
- and $h(\varepsilon) = 0$ by def.
- and $0 \leq 2^0 - 1 = 0$

$P(\alpha x)$: Assume $P(t_1)$, $P(t_2)$.

- WTS $n(\alpha x) \leq 2h(x) - 1$
and \( 0 \leq 2^0 - 1 = 0 \)

\[
P(q_{t_1, b}) \text{ assume } P(t_1), P(t_2).
\]

WTS \( n(q_{t_1, b}) \leq 2^h(q_{t_1, b}) - 1 \)

well, \( n(q_{t_1, b}) = 1 + n(t_1) + n(t_2) \) by def

\[
\leq 1 + \left( 2^{h(t_1) - 1} \right) + n(t_2) \text{ by } P(t_1)
\]

\[
\leq 1 + \left( 2^{h(t_1) - 1} \right) + \left( 2^{h(t_2) - 1} \right) \text{ by } P(t_1)
\]

\[
= 2^{\frac{h(t_1)}{2}} + 2^{\frac{h(t_2)}{2}} - 1
\]

because

\[
\text{max}(h(t_1), h(t_2)) \\
\text{max}(h(t_1), h(t_2)) + 1
\]

\[
= 2^{\frac{h(q_{t_1, b})}{2}} - 1 \text{ by def of } h.
\]