Lecture 9: Countability

- Def$^{\underline{n}}$ of countable
- Countable sets
- Uncountable set

$|A| \leq |B|$ if $\exists$ an inj $f: A \to B$

$|A| \geq |B|$ if $\exists$ a surj: $f: A \to B$

$|A| = |B|$ if $\exists$ a bij $f: A \to B$

Not Cantor-Schroeder-Bernstein:

if f is bij, then f has 2-sided inv.

Actually Cantor-Schroeder-Bernstein:

if $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$

$f: A \to B$ inj $\quad g: B \to A$ inj $\quad \exists\, h: A \to B$ bij.

<u>Def$^{\underline{n}}$</u>: A set X is <u>countable</u> (ctbl) if $|X| \leq |\mathbb{N}|$.

<u>Eqv</u>: • X is ctbl if $|\mathbb{N}| \geq |X|$

• $\exists f : \mathbb{N} \to X$ that is <u>surj.</u>

n	$f(n)$
0	x_0
1	x_1
2	x_2
3	$\vdots$
4	
$\vdots$	

all elts of X are in list (since f surj).

let $X = \mathbb{N} \cup \{-1\}$

<u>Claim:</u> $|X| = |\mathbb{N}|$

$$\begin{array}{ccc} \mathbb{N} & & X \\ 0 & \longleftrightarrow & -1 \\ 1 & \longleftrightarrow & 0 \\ 2 & \longleftrightarrow & 1 \\ \vdots & \longleftrightarrow & 2 \\ & & \vdots \end{array}$$

$\xrightarrow[f]{}$ is a bij.

let $f: \mathbb{N} \to X$ be given by $f(n) := n - 1$.

Check: f is bij.

let $X = \{2n \mid n \in \mathbb{N}\}$

<u>Claim:</u> $|X| = |\mathbb{N}|$

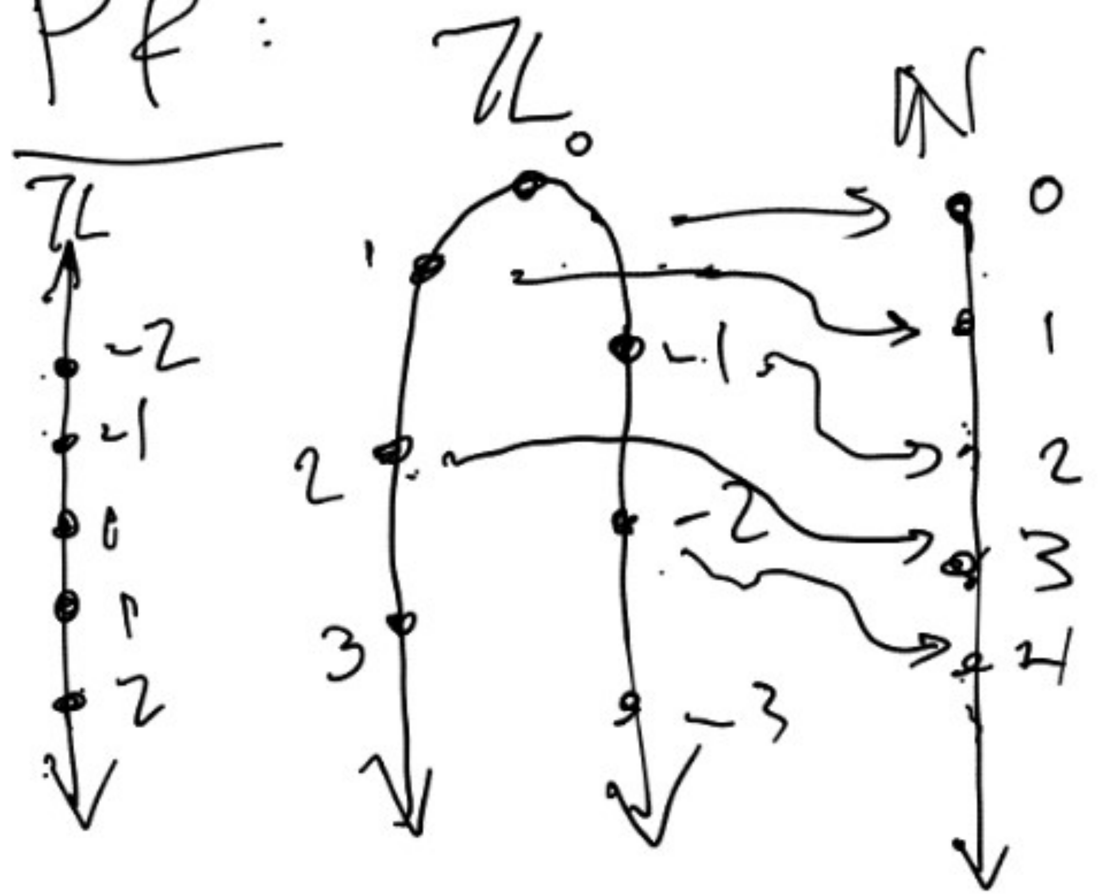
<u>Pf:</u>

X		$\mathbb{N}$
0	$\longrightarrow$	0
2	$\longrightarrow$	1
4	$\longrightarrow$	2
6	$\longrightarrow$	3
8	$\longrightarrow$	4

let $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$

<u>Claim:</u> $|\mathbb{Z}| = |\mathbb{N}|$

<u>Pf:</u>



let $f : \mathbb{Z} \to \mathbb{N}$

be

$$f(i) := \begin{cases} -2i & \text{if } i \leq 0 \\ 2i-1 & \text{if } i > 0 \end{cases}$$

· let $X := \mathbb{N} \times \mathbb{N}$

<u>Claim</u>: $|X| = |\mathbb{N}|$.

<u>Pf</u>:

	0	1	2	3	4 ...
0	$(0,0)^0$	$(0,1)^2$	$(0,2)^5$	... 9	
1	$(1,0)^1$	$(1,1)^4$	$(1,2)^8$ ...		
2	$(2,0)^3$	$(2,1)^7$			
3	⋮				
4					
⋮					

let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be given by

Claim: f is (clearly) surj.

so $|\mathbb{N}| \geq |\mathbb{N} \times \mathbb{N}|$.

let $X = 2^{\mathbb{N}}$,

claim: $|X| \nleq |\mathbb{N}|$ (i.e X is unctbl, i.e $|\mathbb{N}| \ngeq |X|$).

pf: By contra. Assume X is ctbl, i.e.

$\exists f: \mathbb{N} \to X$ that is surj.

ex.

n	$f(n)$	$i \in f(n)$: 0	1	2	3	4	5
0	$\emptyset$	no	no	no	no	...	
1	$\mathbb{N}$	yes	yes	yes	...		
2	evens	yes	no	yes	no	...	
3	mults of 3	yes	no	no	yes	...	
$\vdots$	$\vdots$						
	S_D	yes	no	no	no	...	

let $S_D := \{ i \mid \underset{\in \mathbb{N}}{i} \notin f(i) \}$. "flip everything".

then $S_D \neq f(k)$ for any k.

either ① $k \in f(k)$ or not ② $k \notin f(k)$)

in case ①, $k \in f(k)$, so $k \notin S_D$. So $S_D \neq f(k)$.

in case ②, $k \notin f(k)$, so $k \in S_D$, so $S_D \neq f(k)$.

in either case, $S_D \neq f(k)$, so $S_D \notin \operatorname{im}(f)$ (image)

so f is not surj., a contradiction.

Style of proof is called diagonalization.

0	$\emptyset$	no	no	no	no
1	S_D	yes	no	no	no
2	$\mathbb{N}$	yes	yes	yes	
3	evens				
4	mults of 3				
	S_D'	yes	yes	no	...

let $\mathbb{R}$ = set of real #s.

<u>Claim</u>: $\mathbb{R}$ is unctbl.

<u>pf</u>: for contra, assume $\mathbb{R}$ ctbl,

so $\exists f : \mathbb{N} \to \mathbb{R}$ surj.

ex.

n	$f(n)$	
0	π	3.14159 —
1	0	0.0000 ...
2	1.121212	1.1212 ...
3	1.	1.0000
x_D		8.575 ...

add 5 to each dig.