Lecture 37: Three probability bounds for random variables

- Warmup: Facts about variance, pfs
  - $\text{Var}(X+Y)$, $\text{Var}(cX)$
- Markov's inequality, Chebyshev's inequality, Weak law of big numbers
  - Overview, intuition, proofs

Applications:
- If the expected output of an algorithm is "correct", what is the prob. that the actual output is "very wrong"?
- How to measure averages by sampling
Claim: If $X, Y$ are indep. then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

Pf:

$$\text{Var}(X+Y) = E((X+Y)^2) - (E(X+Y))^2$$

$$= E(X^2 + 2XY + Y^2) - (E(X)+E(Y))^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$

$$= E(X^2) - (E(X))^2 + E(Y^2) - E(Y)^2 + \frac{2E(XY) - 2E(X)E(Y)}{2E(X)E(Y)}$$

$$= \text{Var}(X) + \text{Var}(Y) \checkmark$$

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Claim: $\text{Var}(cX) = c^2 \text{Var}(X)$

Pf: $\text{Var}(cX) = E((cX - E(cX))^2)$

$$= E(c^2(X - E(X))^2)$$

$$= c^2 E((X - E(X))^2) = c^2 \text{Var}(X)$$

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Defn: the standard deviation of $X$ is $\sqrt{\text{Var}(X)}$ (sometimes use $\sigma$ or $\text{StdDev}(X)$)
Markov's inequality:
if \( X \geq 0 \)

\[
\Pr \left( \frac{X}{a} \geq 1 \right) \leq \frac{E(X)}{a}
\]

Chebyshev's inequality:

\[
\Pr \left( |X - E(X)| \geq \alpha \right) \leq \frac{\text{Var}(X)}{\alpha^2}
\]

Weak law of large numbers:

\[
\Pr \left( \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \alpha \right) \leq \frac{\sigma^2}{\alpha^2 n}
\]

if \( X_i \) are RVs
\( E(X_i) = \mu \)
\( \text{Var}(X_i) = \sigma^2 \)
\( X_i \) are independent
Claim (Markov's): \( \Pr(X \geq a) \leq \frac{E(X)}{a} \) if \( X \geq 0 \)

Pf:
\[
E(X) = \sum_{x \in \mathbb{R}} x \cdot \Pr(X = x)
\]
\[
= \sum_{x < 0} x \cdot \Pr(X = x) + \sum_{0 \leq x < a} x \cdot \Pr(X = x) + \sum_{x \geq a} x \cdot \Pr(X = x)
\]

because \( X \geq 0 \)
\[
\geq \sum_{x \geq a} x \cdot \Pr(X = x) \geq \sum_{x \geq a} a \cdot \Pr(X = x) = a \cdot \sum_{x \geq a} \Pr(X = x)
\]
\[
= a \cdot \Pr(X \geq a)
\]

So \( \frac{E(X)}{a} \geq \Pr(X \geq a) \) \( \checkmark \)
Claim (Chebyshev): \[ \Pr \left( |X - E(X)| \geq a \right) \leq \frac{\text{Var}(X)}{a^2} \]

Proof:
\[ \Pr \left( |X - E(X)| \geq a \right) = \Pr \left( (X - E(X))^2 \geq a^2 \right) \]

\[ \leq \frac{E((X - E(X))^2)}{a^2} \]

by Markov's.

\[ = \frac{\text{Var}(X)}{a^2} \]