Lecture 6: Functions

- Define, examples, non-examples

Applications:
- In 2800 everywhere: some examples:
  - Use to define "size" of a set, handling duplicates, infinite sets
  - Use to define "random variables"
- Elsewhere:
  - Functional programming
  - Defining infinite precision numbers
  - Optimizing programs
Def: A function \( f \) from a set \( A \) to a set \( B \) \((f: A \to B)\) is a rule that, for every input \( x \in A \) gives an unambiguous output \( f(x) \in B \).

**Terminology:**
- \( A \) is the **domain** of \( f \), \( \text{dom}(f) \)
- \( B \) is the **codomain** of \( f \), \( \text{cod}(f) \)

**Ex:**
- let \( f(x) = x^2 \) ?
- let \( f: \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^2 \)

**To give a function,** give:
- **domain**
- **codomain**
- **rule**

**Ex:**
- \( \begin{array}{ccc}
  a & \rightarrow & 1 \\
  b & \rightarrow & 2 \\
  c & \rightarrow & 3 \\
\end{array} \)

- **Domain:** \( \{a, b, c\} \)
- **Codomain:** \( \{1, 2, 3\} \)
- **Rule:**
  - \( f(a) = 1 \)
  - \( f(b) = 3 \)
  - \( f(c) = 2 \)

**Partial \( f: \) like a \( f: \) but \( f(x) \) might be undefined for some \( x \).**

**Ex:**
- \( A = \{a, b, c\} \), \( B = \{1, 2, 3\} \)
  - \( a \rightarrow 2 \)
  - \( b \rightarrow 2 \)
  - \( c \rightarrow ? \)

**US**
- **Function**
- **Not a function**

**MCS**
- **Partial**
- **Not a function**

\( (\text{standard math}) \) Not a \( f: \) (lots of CS) Not a \( f: \) (lots of CS)
Example:

- f:
  - a → 1
  - b → 2
  - c → 4

Image of f.

**Definition:**

The image of f (sometimes written Im(f)) is the set of all points that are output by f.

\[
\text{Im}(f) = \{ y \in B \mid \exists x \in A \text{ with } f(x) = y \}.
\]

Why specify a codomain different from image?

Let \( f(x) = \frac{(x-5)^2}{2} \). (f: \( \mathbb{R} \rightarrow \mathbb{R} \))

Codomain: Clearly \( \mathbb{R} \), function obviously outputs \( \mathbb{R} \)

have to think to figure out image

Note: Some people say "range" and mean "image," others mean "codomain." I avoid "range."
\begin{align*}
\begin{array}{c|c}
(a, b, c) & f(c) = 3 \text{?} \\
\hline
(2, 3) & f(c) = 2 \text{? (ambiguous)} \\
(3) & f(c) = 3 \text{?}
\end{array}
\end{align*}

is a relation (we'll see later).

\begin{align*}
\begin{array}{c|c}
K = & f(c) \\
\hline
9 & \text{?} \\
5 & b \text{?} \\
4 & c \text{?}
\end{array}
\end{align*}

\text{Dom: } \{a, b, c\} \\
\text{Cod: } \{1, 2, 3\} \\
\text{Not specified, } \\
N, \text{NC, } - \text{ not a list.}

OK, if cod. is specified.

Can F's represent lists?

\[ \left[ 1, 7, 4, "\text{hello}" \right] \]

Could be represented as a F:

\[ l : \{0, 1, 2, 3\} \rightarrow \{1, 7, 4, "\text{hello}"\} \]

\( l(0) = 1 \quad l(3) = "\text{hello}". \)

\text{(finite) Sequences:}

A sequence can be thought of as a \( \in \text{fin from } N \text{ to } X \)

\[ x_0, x_1, x_2, x_3, \ldots \]

\text{E.g. } (1, 2, 3, 4, 5, \ldots)

Think of as \( x : N \rightarrow N \)

\[ x(1) = 2 \quad x(i) = x_i \]
**Defn: Function f and g are equal if they have the same domain A, codomain B, and for all x ∈ A, \( f(x) = g(x) \).**

**Note:** if \( f = g \) then \( \text{Im}(f) = \text{Im}(g) \).

**(Claim):**

**Proof:** Assume \( f = g \). WTS \( \forall y \in \text{Im}(f), y \in \text{Im}(g) \) and \( \forall y \in \text{Im}(g), y \in \text{Im}(f) \).

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**Note:** Functions are objects like any other, can talk about pairs of \( f \)'s or sets of \( f \)'s or \( f \)'s that input or output other \( f \)'s.

**Defn:** if \( X, Y \) are sets, the \([X \rightarrow Y]\) refers to the set of all \( f \)'s from \( X \) to \( Y \).

**Ex:** \( X = \{a, b\} \quad Y = \{1, 2\} \)

\([X \rightarrow Y]\) = \{ \begin{align*} &a \rightarrow 1 \\ &b \rightarrow 2 \\ &b \rightarrow 2 \end{align*} \}