GRAPH ALGORITHMS

Lecture 19

CS 2110 — Spring 2019

JavaHyperText Topics

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"Graphs", topics:

- 4: DAGs, topological sort
- 5: Planarity
- 6: Graph coloring

Announcements

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Monday after Spring Break there will be a CMS quiz about "Shortest Path" tab of JavaHyperText. To prepare:

- Watch the videos (< 15 min) and their associated PDFs (in total 5 pages)
- Especially try to understand the loop invariant and the development of the algorithm

Announcements

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- Yesterday, 2018 Turing Award Winners announced
- Won for **deep learning** with **neural networks**
  - Facial recognition
  - Talking digital assistants
  - Warehouse robots
  - Self-driving cars
  - …see NYTimes article

Neural networks are graphs!

Neural Network

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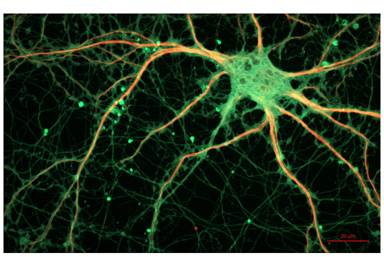

Neurons in brain receive input, maybe fire and activate other neurons

Neural Network

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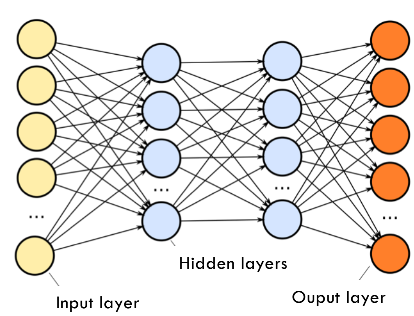

7 Sorting

CS core course prerequisites *(simplified)*

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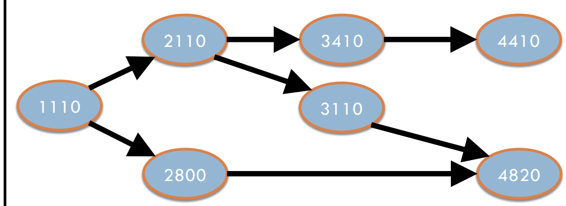

Problem: find an order in which you can take courses without violating prerequisites

e.g. 1110, 2110, 2800, 3110, 3410, 4410, 4820

Topological order

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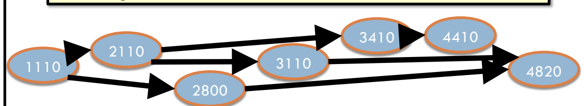
A **topological order** of directed graph G is an ordering of its vertices as $v_1, v_2, \ldots, v_n$, such that for every edge (v_i, v_j), it holds that $i < j$.

Intuition: line up the vertices with all edges pointing left to right.



Cycles

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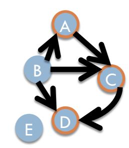
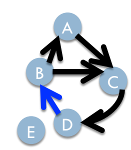
- A directed graph can be topologically ordered if and only if it has no cycles
- A **cycle** is a path $v_0, v_1, \ldots, v_p$ such that $v_0 = v_p$
- A graph is **acyclic** if it has no cycles
- A <u>d</u>irected <u>a</u>cyclic <u>g</u>raph is a **DAG**



DAG



Not a DAG

Is this graph a DAG?

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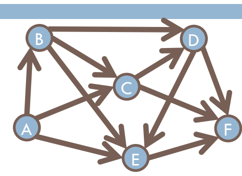

Yes!
It was a DAG.

- Deleting a vertex with indegree zero would not remove any cycles
- Keep deleting such vertices and see whether graph "disappears"

And the order in which we removed vertices was a topological order!

Algorithm: topological sort

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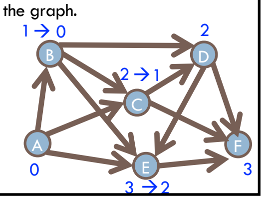
```
k= 0;
// inv: k nodes have been given numbers in 1..k in such a way that
        if n1 <= n2, there is no edge from n2 to n1.
while (there is a node of in-degree 0) {
    Let n be a node of in-degree 0;
    Give it number k;
    Delete n and all edges leaving it from the graph.
    k= k+1;
}
```

k= 0 → 1

A 0
B 1
C
D
E
F

JavaHyperText shows how to implement efficiently: O(V+E) running time.



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Graph Coloring

Map coloring

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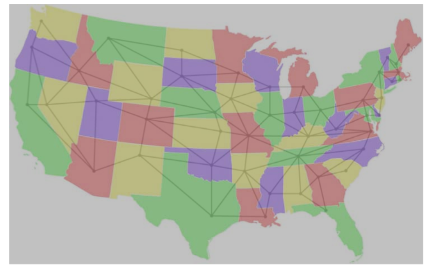
How many colors are needed to ensure adjacent states have different colors?



Graph coloring

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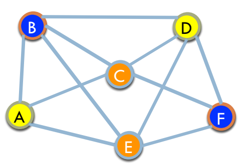
Coloring: assignment of color to each vertex.
Adjacent vertices must have different colors.



How many colors needed?

Uses of graph coloring

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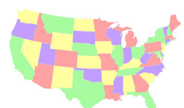

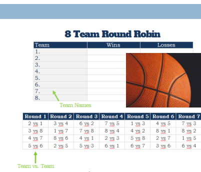

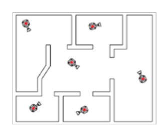

And more! http://ijcit.org/ijcit_papers/vol3no2/IJCIT-130101.pdf

How to color a graph

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```
void color() {
  for each vertex v in graph:
    c= find_color(neighbors of v);
    color v with c;
}
int find_color(vs) {
  int[] used;
  assign used[c] the number of vertices in vs that are colored c


  return smallest c such that used[c] == 0;
}
```

Assume colors are integers 0, 1, …

How to color a graph

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```
void color() {
  for each vertex v in graph:
    c= find_color(neighbors of v);
    color v with c;
}
int find_color(vs) {
  int[] used= new int[vs.length() + 1];
  for each vertex v in vs:
    if color(v) <= vs.length():
      used[color(v)]++;
  }
  return smallest c such that used[c] == 0;
}
```

Assume colors are integers 0, 1, …

If there are d vertices, need at most d+1 available colors

Analysis

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```
void color() {
  for each vertex v in graph:
    c= find_color(neighbors of v);
    color v with c;
}
int find_color(vs) {
  int[] used= new int[vs.length() + 1];
  for each vertex v in vs:
    if color(v) <= vs.length():
      used[color(v)]++;
  }
  return smallest c such that used[c] == 0;
}
```

Total time: O(E)

Time: O(# neighbors of v)

Time: O(vs.length())

Analysis

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```
void color() {
  for each vertex v in graph:
    c= find_color(neighbors of v);
    color v with c;
}
int find_color(vs) {
  int[] used= new int[vs.length() + 1];
  for each vertex v in vs:
    if color(v) <= vs.length():
      used[color(v)]++;
  }
  return smallest c such that used[c] == 0;
}
```

Use the minimum number of colors?

Maybe! Depends on order vertices processed.

Analysis

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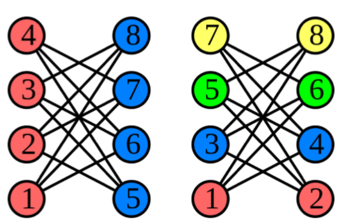
Best coloring Worst coloring



Vertices labeled in order of processing

Only 2 colors needed for this special kind of graph…

Source: https://en.wikipedia.org/wiki/Grundy_number#/media/File:Greedy_colourings.svg

Bipartite graphs

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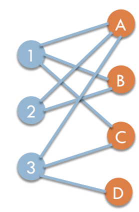
Bipartite: vertices can be partitioned into two sets such that no edge connects two vertices in the same set



Matching problems:

- Med students & hospital residencies
- TAs to discussion sections
- Football players to teams

Fact: G is bipartite iff G is 2-colorable

Four Color Theorem

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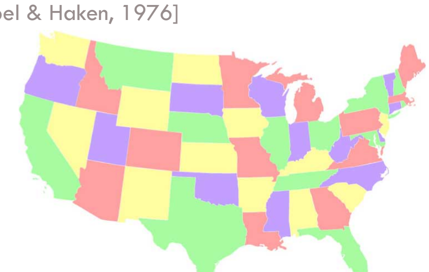
Every "map-like" graph is 4-colorable

[Appel & Haken, 1976]



Four Color Theorem

Proof required checking that 1,936 special graphs had a certain property

- Appel & Haken used a computer program to check the 1,936 graphs
- Does that count as a proof?
- Gries looked at their computer program and found an error; it could be fixed

In 2008 entire proof formalized in Coq proof assistant [Gonthier & Werner]: see CS 4160

Four Color Theorem

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Every "map-like" graph is 4-colorable

[Appel & Haken, 1976]

…"map-like"?

= planar

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Planar Graphs

Planarity

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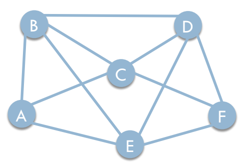
A graph is **planar** if it can be drawn in the plane without any edges crossing



Discuss: Is this graph planar?

Planarity

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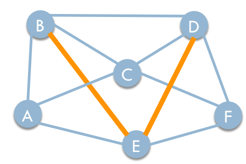
A graph is **planar** if it can be drawn in the plane without any edges crossing



Discuss: Is this graph planar?

Planarity

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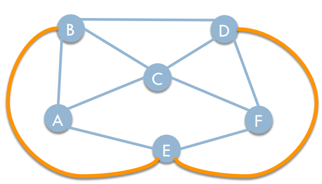
A graph is **planar** if it can be drawn in the plane without any edges crossing



Discuss: Is this graph planar?

YES!

Detecting Planarity

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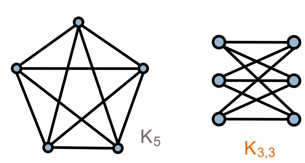
Kuratowski's Theorem:



A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown).

John Hopcroft & Robert Tarjan

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- **Turing Award in 1986** "for fundamental achievements in the design and analysis of algorithms and data structures"

- One of their fundamental achievements was a O(V) algorithm for determining whether a graph is planar.

David Gries & Jinyun Xue

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Tech Report, 1988

Abstract: We give a rigorous, yet, we hope, readable, presentation of the Hopcroft-Tarjan linear algorithm for testing the planarity of a graph, using more modern principles and techniques for developing and presenting algorithms that have been developed in the past 10-12 years (their algorithm appeared in the early 1970's). Our algorithm not only tests planarity but also constructs a planar embedding, and in a fairly straightforward manner. The paper concludes with a short discussion of the advantages of our approach.