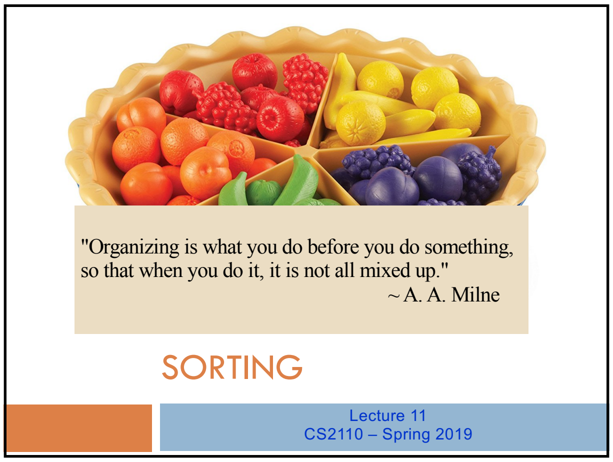
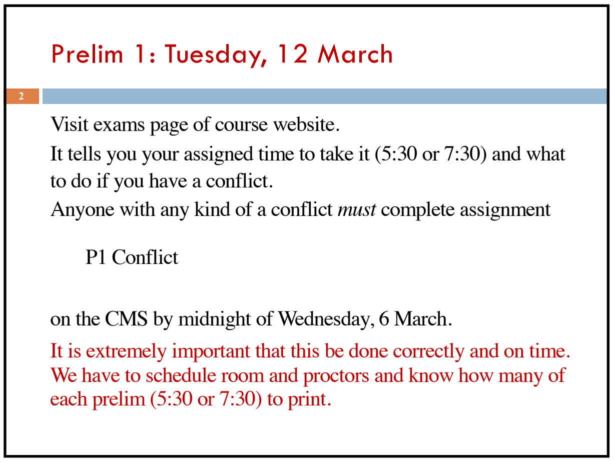


"Organizing is what you do before you do something, so that when you do it, it is not all mixed up."
~ A. A. Milne

SORTING

Lecture 11
CS2110 – Spring 2019



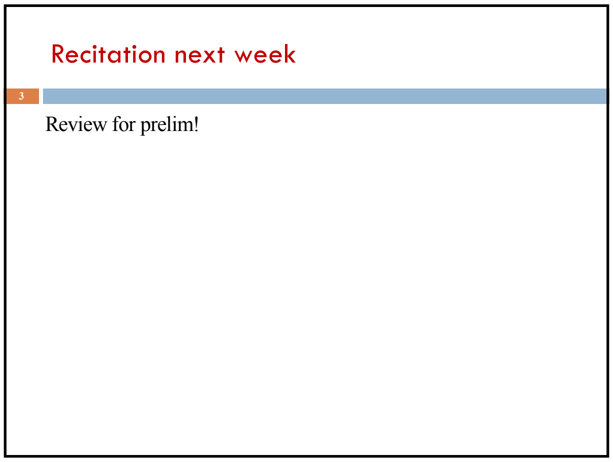
Prelim 1: Tuesday, 12 March

2

Visit exams page of course website.
It tells you your assigned time to take it (5:30 or 7:30) and what to do if you have a conflict.
Anyone with any kind of a conflict *must* complete assignment

P1 Conflict

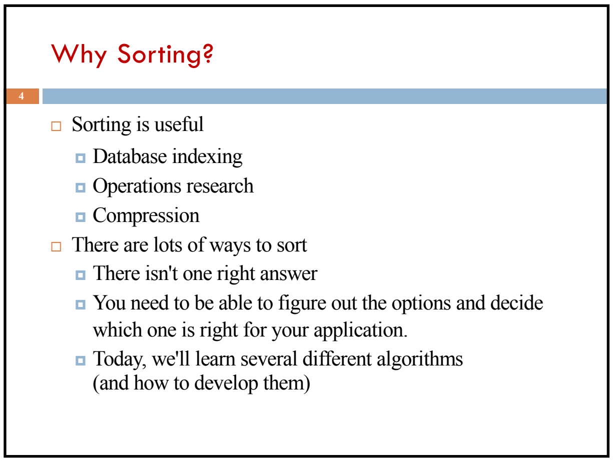
on the CMS by midnight of Wednesday, 6 March.
It is extremely important that this be done correctly and on time. We have to schedule room and proctors and know how many of each prelim (5:30 or 7:30) to print.



Recitation next week

3

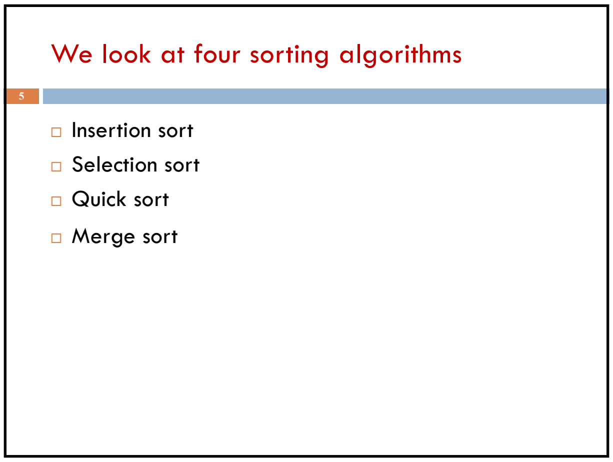
Review for prelim!



Why Sorting?

4

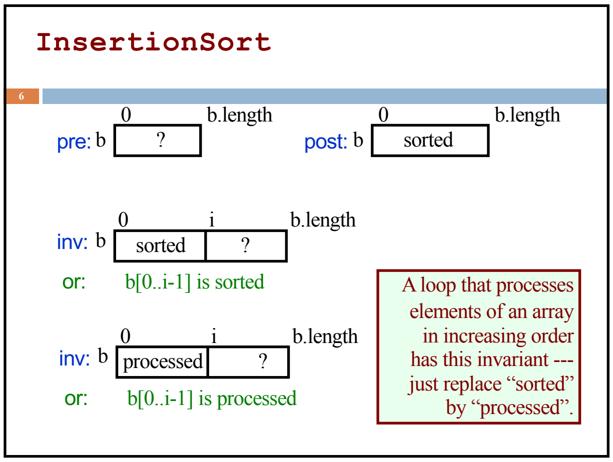
- □ Sorting is useful
  - ◘ Database indexing
  - ◘ Operations research
  - ◘ Compression
- □ There are lots of ways to sort
  - ◘ There isn't one right answer
  - ◘ You need to be able to figure out the options and decide which one is right for your application.
  - ◘ Today, we'll learn several different algorithms (and how to develop them)



We look at four sorting algorithms

5

- □ Insertion sort
- □ Selection sort
- □ Quick sort
- □ Merge sort



InsertionSort

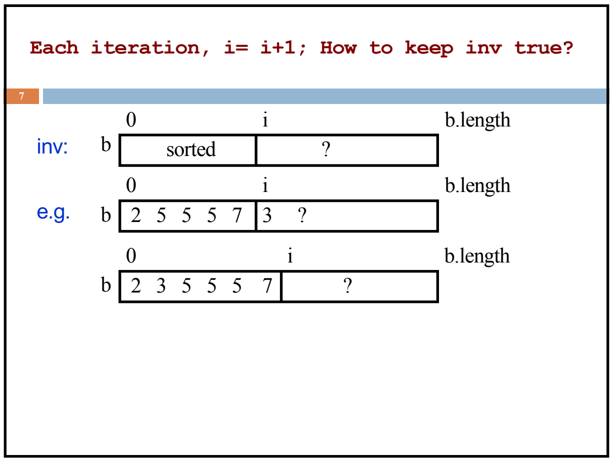
6

pre: b [0 ? b.length] post: b [0 sorted b.length]

inv: b [0 sorted i ? b.length]
or: b[0..i-1] is sorted

inv: b [0 processed i ? b.length]
or: b[0..i-1] is processed

A loop that processes elements of an array in increasing order has this invariant --- just replace "sorted" by "processed".

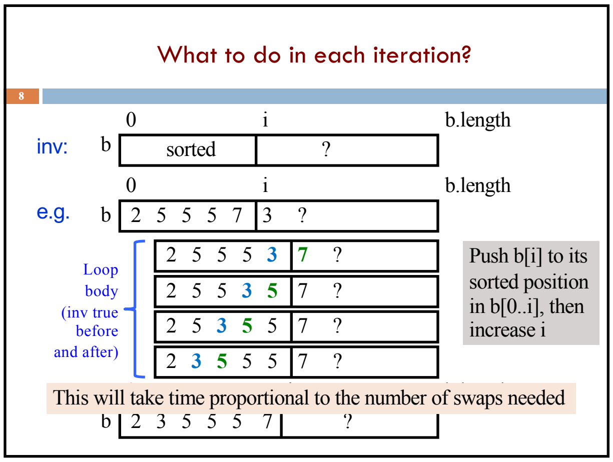


Each iteration, i= i+1; How to keep inv true?

inv: b
0 | i | b.length
sorted | ?

e.g. b
0 | i | b.length
2 5 5 5 7 | 3 ?

b
0 | i | b.length
2 3 5 5 5 7 | ?



What to do in each iteration?

inv: b
0 | i | b.length
sorted | ?

e.g. b
0 | i | b.length
2 5 5 5 7 | 3 ?

Loop body (inv true before and after)

2 5 5 5 3 | 7 ?

2 5 5 3 5 | 7 ?

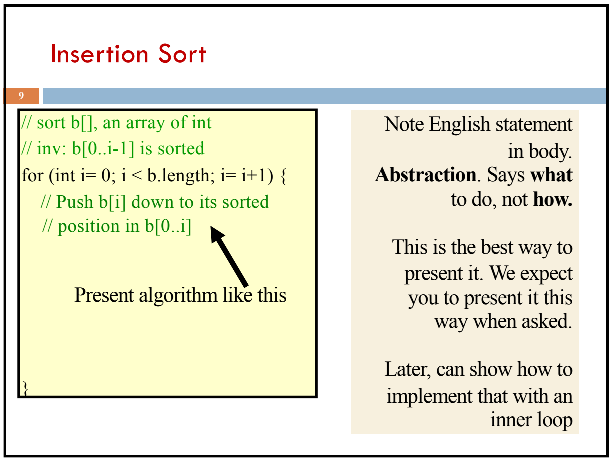
2 5 3 5 5 | 7 ?

2 3 5 5 5 | 7 ?

Push b[i] to its sorted position in b[0..i], then increase i

This will take time proportional to the number of swaps needed

b | 2 3 5 5 5 7 | ?



Insertion Sort

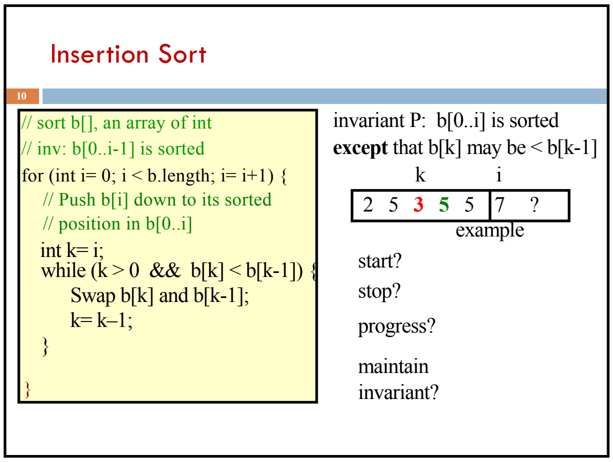
```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
    // Push b[i] down to its sorted
    // position in b[0..i]

}
```

Present algorithm like this

Note English statement in body. **Abstraction**. Says **what** to do, not **how.**

This is the best way to present it. We expect you to present it this way when asked.

Later, can show how to implement that with an inner loop



Insertion Sort

```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
    // Push b[i] down to its sorted
    // position in b[0..i]
    int k= i;
    while (k > 0  &&  b[k] < b[k-1]) {
        Swap b[k] and b[k-1];
        k= k–1;
    }
}
```

invariant P: b[0..i] is sorted **except** that b[k] may be < b[k-1]

k | i
2 5 3 5 5 | 7 ?

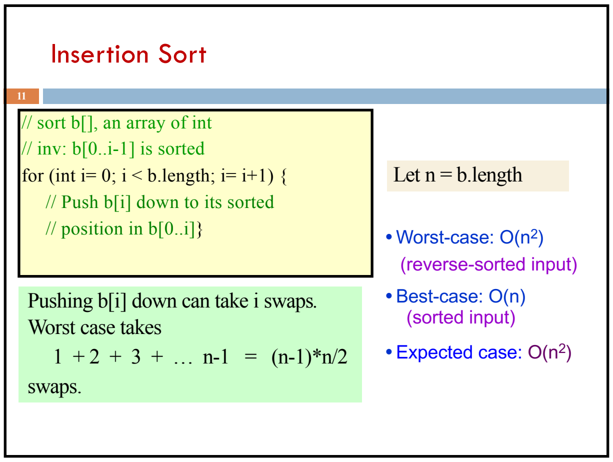
example

start?

stop?

progress?

maintain invariant?



Insertion Sort

```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
    // Push b[i] down to its sorted
    // position in b[0..i]}
```

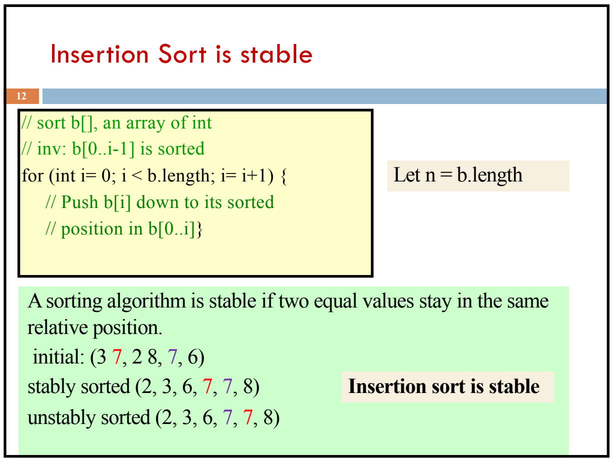
Let n = b.length

- Worst-case: $O(n^2)$ (reverse-sorted input)
- Best-case: $O(n)$ (sorted input)
- Expected case: $O(n^2)$

Pushing b[i] down can take i swaps. Worst case takes

1 + 2 + 3 + … n-1 = (n-1)*n/2

swaps.



Insertion Sort is stable

```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
    // Push b[i] down to its sorted
    // position in b[0..i]}
```

Let n = b.length

A sorting algorithm is stable if two equal values stay in the same relative position.

initial: (3 7, 2 8, 7, 6)

stably sorted (2, 3, 6, 7, 7, 8)

unstably sorted (2, 3, 6, 7, 7, 8)

Insertion sort is stable

Performance

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Algorithm	Ave time. Worst-case time		Space	Stable?
Insertion Sort	$O(n^2)$.	$O(n^2)$	$O(1)$	Yes

SelectionSort

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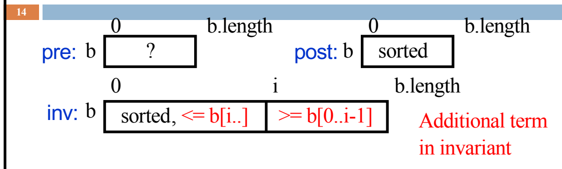
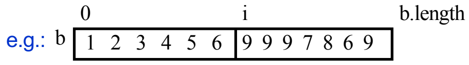


Additional term in invariant

Keep invariant true while making progress?



Increasing i by 1 keeps inv true only if b[i] is min of b[i..]

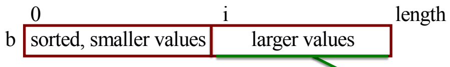
SelectionSort

```
// sort b[].
// inv: b[0..i-1] sorted  AND
//       b[0..i-1]  <=  b[i..]
for (int i= 0; i < b.length; i= i+1) {
   int m= index of min of b[i..];
   Swap b[i] and b[m];
}
```

Another common way for people to sort cards

Runtime
with n = b.length

- Worst-case O(n^2)
- Best-case O(n^2)
- Expected-case O(n^2)

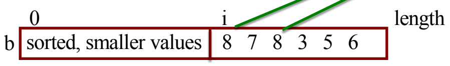


Each iteration, swap min value of this section into b[i]

SelectionSort —not stable

```
// sort b[].
// inv: b[0..i-1] sorted  AND
//       b[0..i-1]  <=  b[i..]
for (int i= 0; i < b.length; i= i+1) {
   int m= index of min of b[i..];
   Swap b[i] and b[m];
}
```

Here, swapping b[i] with the minimum of b[i..], 3, changes the relative position of the two 8's.



Performance

17

Algorithm	Ave time. Worst-case time		Space	Stable?
Insertion sort	$O(n^2)$	$O(n^2)$	$O(1)$	Yes
Selection sort	$O(n^2)$	$O(n^2)$	$O(1)$	No

QuickSort

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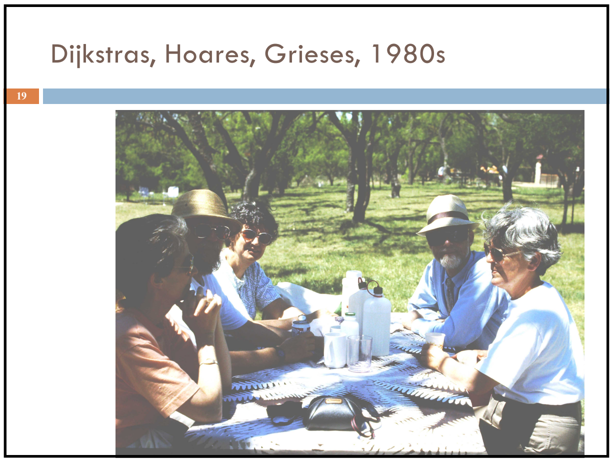
Quicksort developed by Sir Tony Hoare (he was knighted by the Queen of England for his contributions to education and CS).

84 years old.

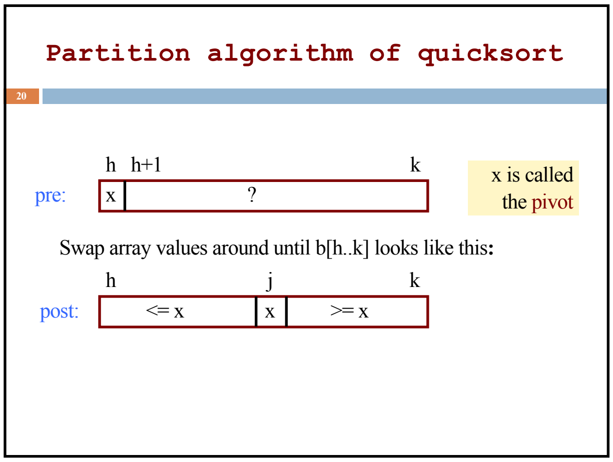
Developed Quicksort in 1958. But he could not explain it to his colleague, so he gave up on it.

Later, he saw a draft of the new language Algol 58 (which became Algol 60). It had recursive procedures. First time in a procedural programming language. "Ah!," he said. "I know how to write it better now." 15 minutes later, his colleague also understood it.



Dijkstras, Hoares, Grieses, 1980s

19



Partition algorithm of quicksort

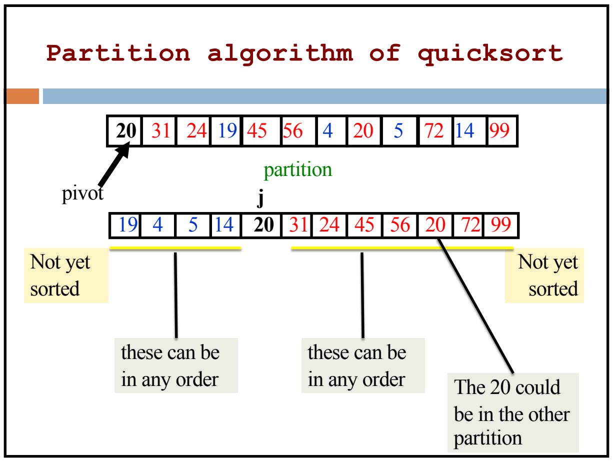
20

pre: h h+1 k: | x | ? |

x is called the pivot

Swap array values around until b[h..k] looks like this:

post: h j k: | <= x | x | >= x |



Partition algorithm of quicksort

20	31	24	19	45	56	4	20	5	72	14	99

pivot

partition

j

19	4	5	14	20	31	24	45	56	20	72	99

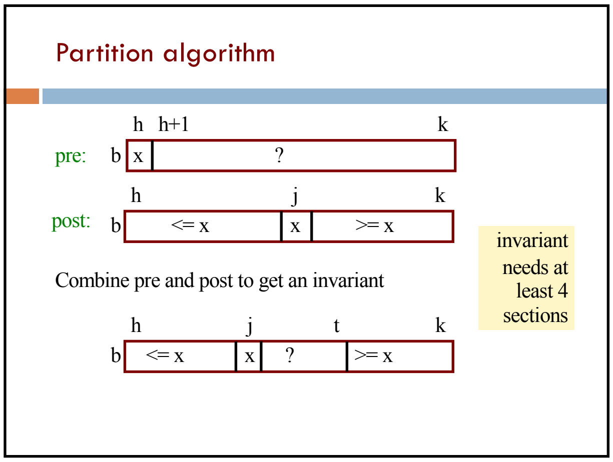
Not yet sorted

Not yet sorted

these can be in any order

these can be in any order

The 20 could be in the other partition



Partition algorithm

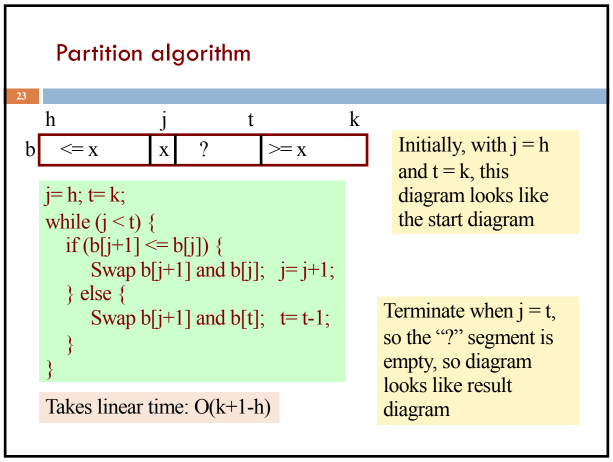
pre: h h+1 k: b | x | ? |

post: h j k: b | <= x | x | >= x |

Combine pre and post to get an invariant

h j t k: b | <= x | x | ? | >= x |

invariant needs at least 4 sections



Partition algorithm

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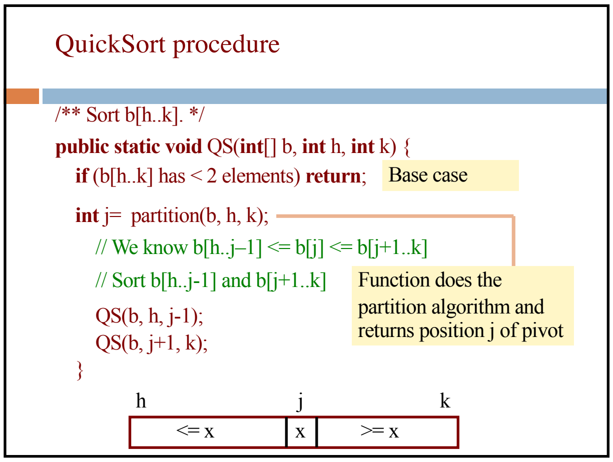
h j t k: b | <= x | x | ? | >= x |

```
j= h; t= k;
while (j < t) {
   if (b[j+1] <= b[j]) {
        Swap b[j+1] and b[j];   j= j+1;
   } else {
        Swap b[j+1] and b[t];   t= t-1;
   }
}
```

Takes linear time: O(k+1-h)

Initially, with j = h and t = k, this diagram looks like the start diagram

Terminate when j = t, so the "?" segment is empty, so diagram looks like result diagram



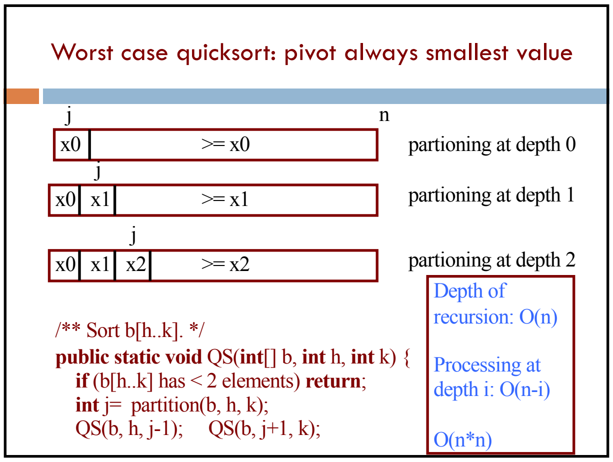
QuickSort procedure

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
  if (b[h..k] has < 2 elements) return;    Base case

  int j= partition(b, h, k);
    // We know b[h..j–1] <= b[j] <= b[j+1..k]
    // Sort b[h..j-1] and b[j+1..k]
    QS(b, h, j-1);
    QS(b, j+1, k);
}
```

Function does the partition algorithm and returns position j of pivot

h j k: | <= x | x | >= x |



Worst case quicksort: pivot always smallest value

j n

| x0 | >= x0 |

partioning at depth 0

j

| x0 | x1 | >= x1 |

partioning at depth 1

j

| x0 | x1 | x2 | >= x2 |

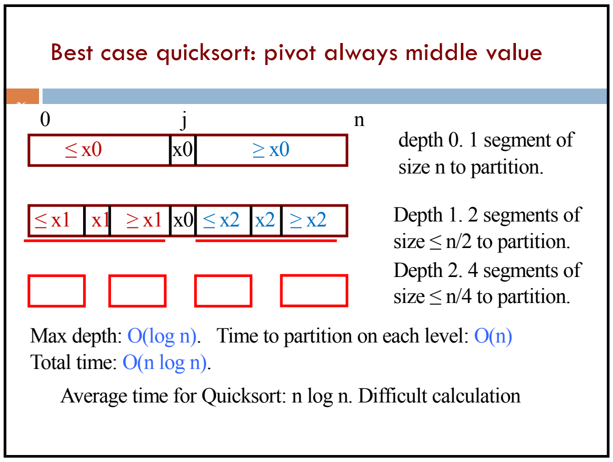
partioning at depth 2

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j=  partition(b, h, k);
   QS(b, h, j-1);     QS(b, j+1, k);
```

Depth of recursion: O(n)

Processing at depth i: O(n-i)

O(n*n)



Best case quicksort: pivot always middle value

0 j n

| ≤ x0 | x0 | ≥ x0 |

depth 0. 1 segment of size n to partition.

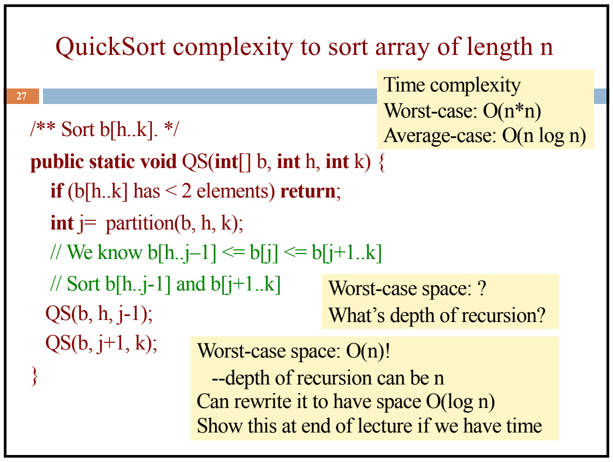
| ≤ x1 | x1 | ≥ x1 | x0 | ≤ x2 | x2 | ≥ x2 |

Depth 1. 2 segments of size ≤ n/2 to partition.

Depth 2. 4 segments of size ≤ n/4 to partition.

Max depth: O(log n). Time to partition on each level: O(n)
Total time: O(n log n).

Average time for Quicksort: n log n. Difficult calculation



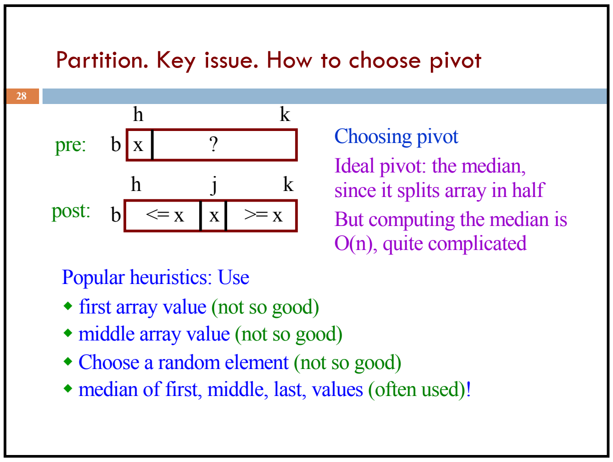
QuickSort complexity to sort array of length n

Time complexity
Worst-case: O(n*n)
Average-case: O(n log n)

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j=  partition(b, h, k);
   // We know b[h..j–1] <= b[j] <= b[j+1..k]
   // Sort b[h..j-1] and b[j+1..k]
  QS(b, h, j-1);
  QS(b, j+1, k);
}
```

Worst-case space: ?
What's depth of recursion?

Worst-case space: O(n)!
  --depth of recursion can be n
Can rewrite it to have space O(log n)
Show this at end of lecture if we have time



Partition. Key issue. How to choose pivot

pre: b

h k

| x | ? |

post: b

h j k

| <= x | x | >= x |

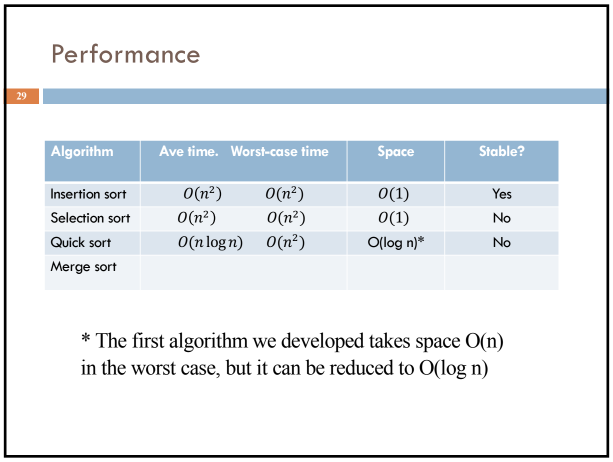
Choosing pivot

Ideal pivot: the median, since it splits array in half

But computing the median is O(n), quite complicated

Popular heuristics: Use

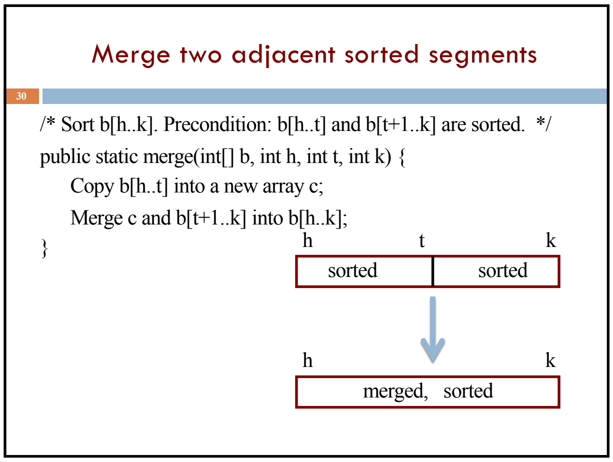
- first array value (not so good)
- middle array value (not so good)
- Choose a random element (not so good)
- median of first, middle, last, values (often used)!



Performance

Algorithm	Ave time.	Worst-case time	Space	Stable?
Insertion sort	$O(n^2)$	$O(n^2)$	$O(1)$	Yes
Selection sort	$O(n^2)$	$O(n^2)$	$O(1)$	No
Quick sort	$O(n\log n)$	$O(n^2)$	O(log n)*	No
Merge sort				

* The first algorithm we developed takes space O(n) in the worst case, but it can be reduced to O(log n)



Merge two adjacent sorted segments

```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted.  */
public static merge(int[] b, int h, int t, int k) {
     Copy b[h..t] into a new array c;
     Merge c and b[t+1..k] into b[h..k];
}
```

h t k

| sorted | sorted |

h k

| merged,   sorted |

Merge two adjacent sorted segments

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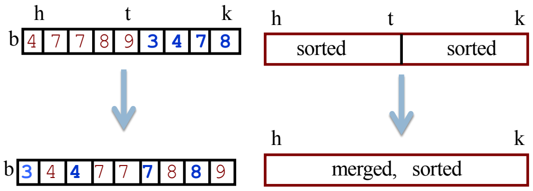
```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted.  */
public static merge(int[] b, int h, int t, int k) {
}
```



```
   h       t       k          h          t          k
 b 4 7 7 8 9 3 4 7 8          |  sorted  |  sorted  |

 b 3 4 4 7 7 7 8 8 9          h                     k
                              |   merged,   sorted  |
```

Merge two adjacent sorted segments

32

```
    0     t
  c 4 7 7 8 9

    h     t       k
  b ? ? ? ? ? 3 4 7 8

    h     t       k
  b 3 4 4 7 7 7 8 8 9
```

Place all values in c[0..] and b[t+1..k] into b[h..k] in sorted order

Merge two adjacent sorted segments

33

```
    0
  c 4 7 7 8 9

    h     t       k
  b ? ? ? ? ? 3 4 7 8
```

Place all values in c[0..] and b[t+1..k] into b[h..k] in sorted order

Merge two adjacent sorted segments

34

```
    0
  c 4 7 7 8 9

    h     t       k
  b 3 ? ? ? ? ? 4 7 8
```

Step 1. Move **3** from b[t+1] to b[h]

Merge two adjacent sorted segments

35

```
    0
  c ? 7 7 8 9

    h     t       k
  b 3 4 ? ? ? ? 4 7 8
```

Step 2. Move 4 from c[0] to b[h+1]

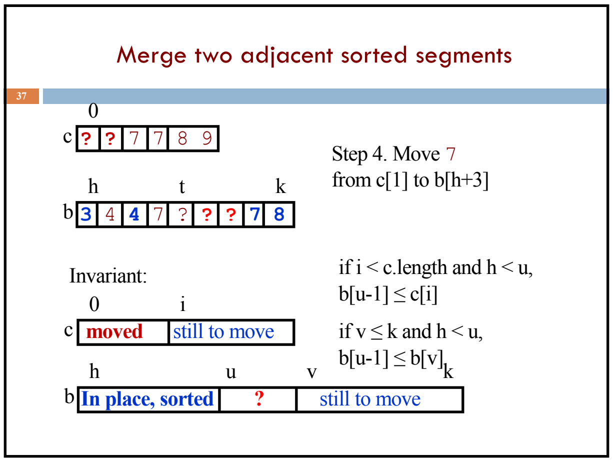
Merge two adjacent sorted segments

36

```
    0
  c ? 7 7 8 9

    h     t       k
  b 3 4 4 ? ? ? ? 7 8
```

Step 3. Move **4** from b[t+2] to b[h+2]



Merge two adjacent sorted segments

37

0

c ? ? 7 7 8 9

Step 4. Move 7
from c[1] to b[h+3]

h t k

b 3 4 4 7 ? ? ? 7 8

Invariant:

0 i

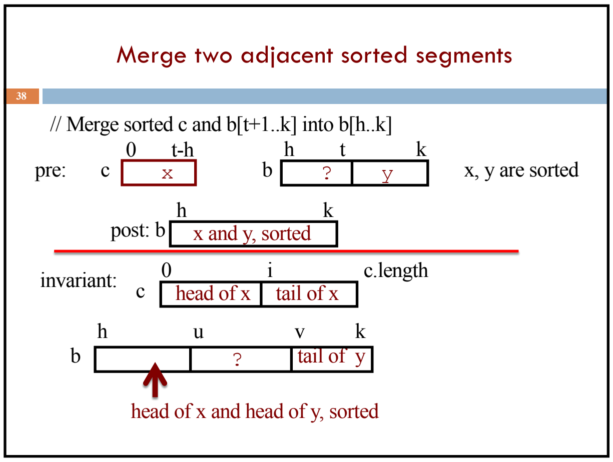
c moved still to move

h u v k

b In place, sorted ? still to move

if i < c.length and h < u,
b[u-1] ≤ c[i]

if v ≤ k and h < u,
b[u-1] ≤ b[v]



Merge two adjacent sorted segments

38

// Merge sorted c and b[t+1..k] into b[h..k]

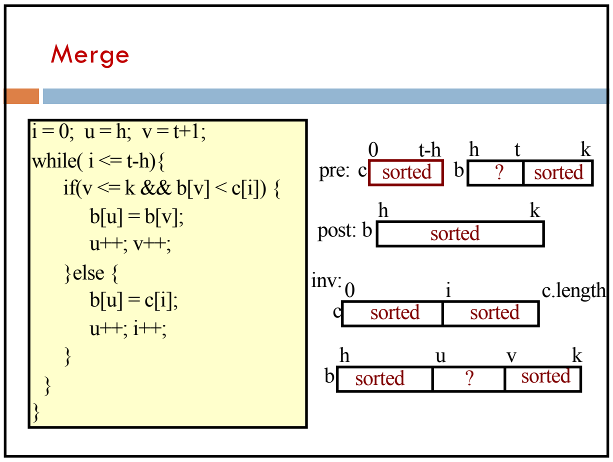
pre: c (0, t-h) x   b (h, t, k) ? y   x, y are sorted

post: b (h, k) x and y, sorted

invariant: c (0, i, c.length) head of x tail of x

b (h, u, v, k) ? tail of y

head of x and head of y, sorted



Merge

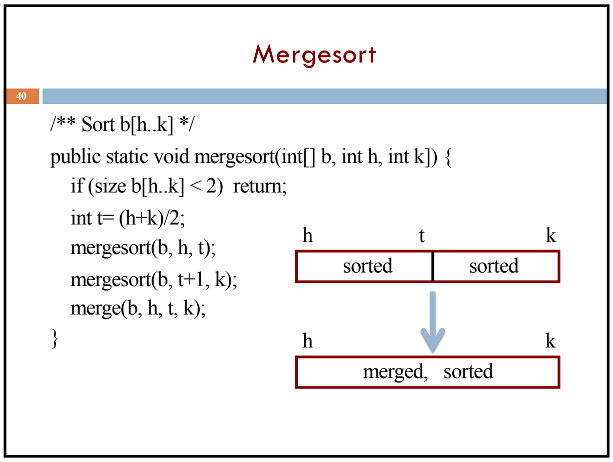
```
i = 0;  u = h;  v = t+1;
while( i <= t-h){
     if(v <= k && b[v] < c[i]) {
          b[u] = b[v];
          u++; v++;
     }else {
          b[u] = c[i];
          u++; i++;
     }
  }
}
```

pre: c (0, t-h) sorted   b (h, t, k) ? sorted

post: b (h, k) sorted

inv: c (0, i, c.length) sorted sorted

b (h, u, v, k) sorted ? sorted



Mergesort

40

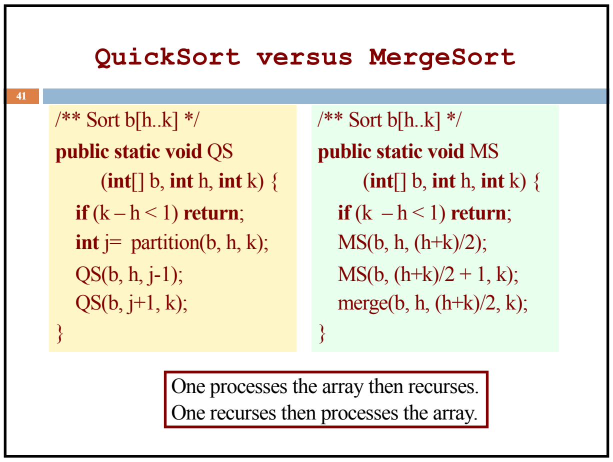
```
/** Sort b[h..k] */
public static void mergesort(int[] b, int h, int k]) {
   if (size b[h..k] < 2)  return;
   int t= (h+k)/2;
   mergesort(b, h, t);
   mergesort(b, t+1, k);
   merge(b, h, t, k);
}
```

h t k

sorted sorted

h k

merged, sorted



QuickSort versus MergeSort

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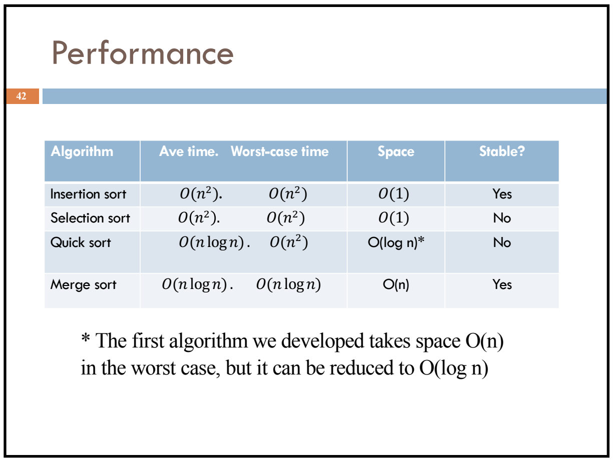
```
/** Sort b[h..k] */
public static void QS
      (int[] b, int h, int k) {
   if (k – h < 1) return;
   int j=  partition(b, h, k);
   QS(b, h, j-1);
   QS(b, j+1, k);
}
```

```
/** Sort b[h..k] */
public static void MS
      (int[] b, int h, int k) {
   if (k  – h < 1) return;
   MS(b, h, (h+k)/2);
   MS(b, (h+k)/2 + 1, k);
   merge(b, h, (h+k)/2, k);
}
```

One processes the array then recurses.
One recurses then processes the array.



Performance

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Algorithm	Ave time. Worst-case time	Space	Stable?
Insertion sort	$O(n^2)$. $O(n^2)$	$O(1)$	Yes
Selection sort	$O(n^2)$. $O(n^2)$	$O(1)$	No
Quick sort	$O(n \log n)$. $O(n^2)$	O(log n)*	No
Merge sort	$O(n \log n)$. $O(n \log n)$	O(n)	Yes

* The first algorithm we developed takes space O(n)
in the worst case, but it can be reduced to O(log n)

Sorting in Java

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- Java.util.Arrays has a method sort(array)
  - implemented as a collection of overloaded methods
  - for primitives, sort is implemented with a version of quicksort
  - for Objects that implement Comparable, sort is implemented with timSort, a modified mergesort developed in 1993 by Tim Peters
- Tradeoff between speed/space and stability/performance guarantees

Quicksort with logarithmic space

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Problem is that if the pivot value is always the smallest (or always the largest), the depth of recursion is the size of the array to sort.

Eliminate this problem by doing some of it iteratively and some recursively. We may show you this later. Not today!

QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        Reduce the size of b[h1..k1], keeping inv true
   }
}
```

QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        int j= partition(b, h1, k1);
        // b[h1..j-1] <= b[j] <= b[j+1..k1]
        if (b[h1..j-1] smaller than b[j+1..k1])
             {  QS(b, h, j-1);  h1= j+1; }
        else
             {QS(b, j+1, k1);  k1= j-1; }
   }
}
```

Only the smaller segment is sorted recursively. If b[h1..k1] has size n, the smaller segment has size < n/2. Therefore, depth of recursion is at most log n