It’s turtles all the way down

RECURSION

Lecture 8

CS2110 – Spring 2019

Recursion: Look at Java Hypertext entry "recursion".

We've covered almost everything in Java! Just a few more things to introduce, which will be covered from time to time.

Assignment A3 is about linked lists. We'll spend 5-10 minutes on it in next Tuesday's lecture.

Note: For next week, the tutorial you have to watch is about loop invariants. We'll introduce it in this lecture. It's important to master this material, because we *use* it a lot in later lectures.

You know about method specifications and class invariants. Now comes the loop invariant.

Next recitation: Loop invariants

In JavaHyperText, click on link Loop invariants in the horizontal navigation bar. Watch the videos on that page and the second page, 2. Practice on developing parts of loops.

There will be a short quiz on Loop invariants and a problem set to do during recitation.

We now introduce the topic.

Next recitation: Loop invariants

// store in s the sum of the elements in array b.

```
int k= 0; s= 0;
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
```

0	1	2	3	4
3	2	5	1	

when done, s = 11

Why start with k = 0?

How do you know that s has the right value when the loop terminates?

Why is b[k] added to s?

Without giving meaning to variables, the only way you can tell this works is by executing it in your head, see what is does on a small array. A loop invariant will give that meaning.

Next recitation: Loop invariants


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

0	1	2	3	4
3	2	5	1	

	0	k	b.length
P: b	s is sum of these	?	

Loopy question 1: Does init truthify P?


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

k

0	1	2	3	4
3	2	5	1	

s $\boxed{0}$

	0	k	b.length
P: b	s is sum of these	?	

Loopy question 2: Is R true upon termination?


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

	0	1	2	3	4
					k
b	3	2	5	1	

s 11

	0	k	b.length
P: b	s is sum of these	?	

Loopy question 3: Does repetend make progress toward termination?


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

k

0	1	2	3	4
3	2	5	1	

s: 5

	0	k	b.length
P: b	s is sum of these	?	

Loopy question 4: Does repetend keep invariant true?


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

k

0	1	2	3	4
3	2	5	1	

s: 5

	0	k	b.length
P: b	s is sum of these	?	

Loopy question 4: Does repetend keep invariant true?


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

This will be true before and after each iteration

			k	
0	1	2	3	4
3	2	5	1	

s $\boxed{10}$

	0	k	b.length
P: b	s is sum of these	?	

All four loopy questions checked.
Loop is correct.


```
int k= 0; s= 0;
// invariant P: s = sum of b[0..k-1]
while (k < b.length) {
    s= s + b[k];
    k= k+1;
}
// R: s = sum of b[0..b.length-1]
```

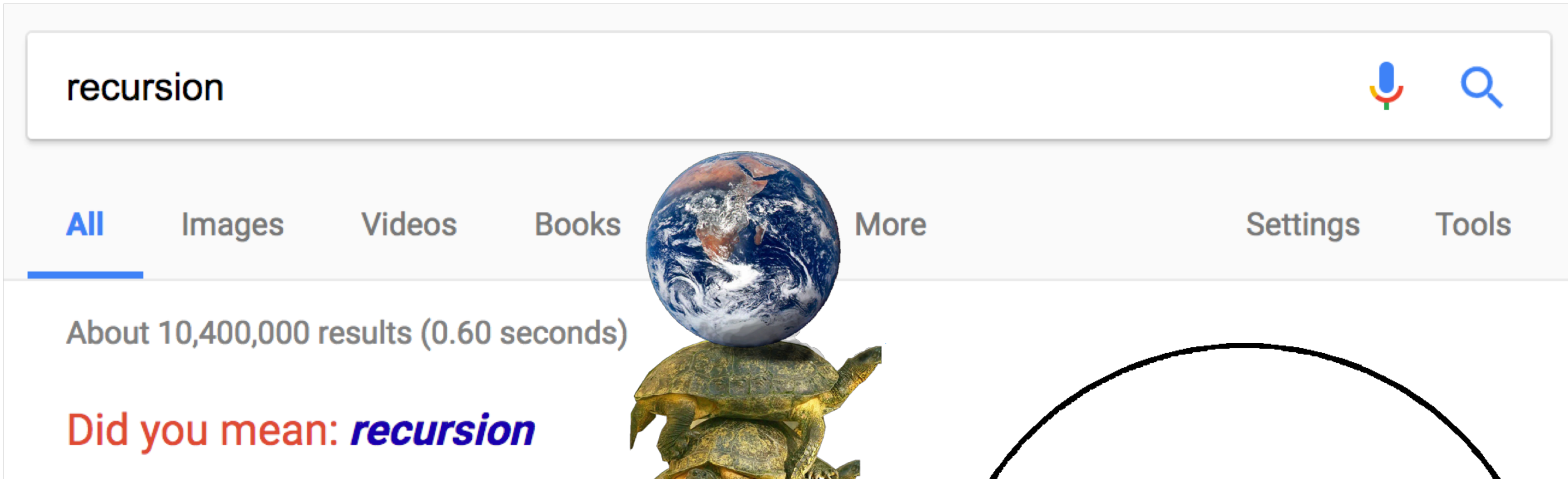
		0	k
P:	b	s is sum of these	?

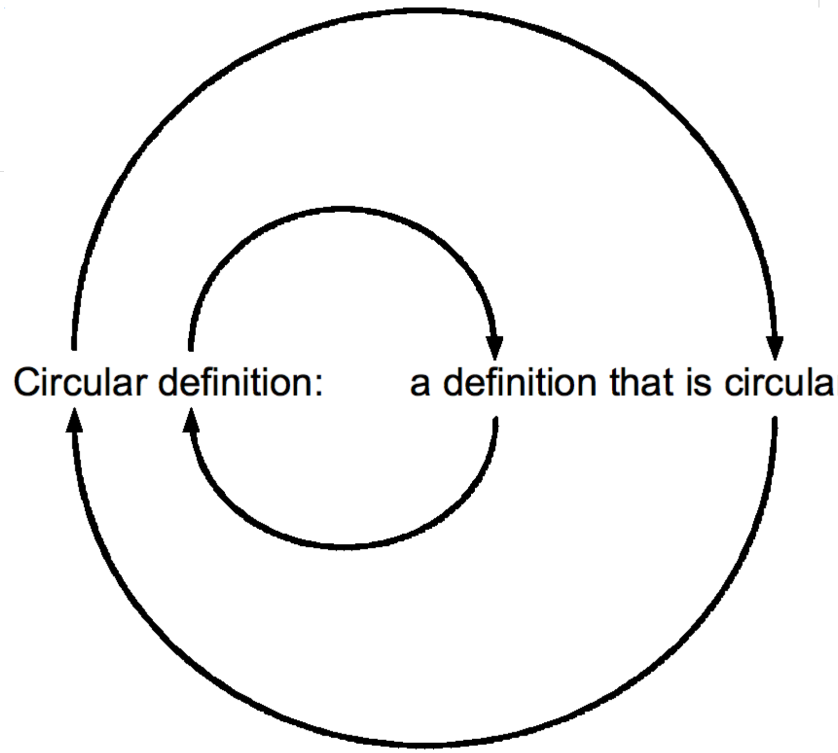
Use of invariant allows us to break loop (and init) into parts and handle them independently.

Initialization? Look only at possible precondition of algorithm and loop invariant

Termination? Look only at loop invariant, loop condition, postcondition.

To Understand Recursion…





Recursion – Real Life Examples

<noun phrase> is <noun>, or

<adjective> <noun phrase>, or

<adverb> <noun phrase>

Example:

terrible horrible no-good very bad day

Recursion – Real Life Examples

<noun phrase> is <noun>, or

<adjective> <noun phrase>, or

<adverb> <noun phrase>

ancestor(p) is parent(p), or

parent(ancestor(p))

great great great great great great great great great great great great great grandmother.

0! = 1

n! = n * (n-1)!

1, 1, 2, 6, 24, 120, 720, 5050, 40320, 362880, 3628800, 39916800, 479001600…

Sum the digits in a non-negative integer


```
/** = sum of digits in n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // { n has at least two digits }
    // return first digit + sum of rest
    return n%10  +  sum(n/10);
}
```

sum calls itself!

sum(7) = 7

sum(8703) = 3 + sum(870)
= 3 + 8 + sum(70)
= 3 + 8 + 7 + sum(0)

Two different questions, two different answers

1. How is it executed?
(or, why does this even work?)

2. How do we understand recursive methods?
(or, how do we write/develop recursive methods?)

Stacks and Queues

stack grows

top element
2nd element
...
bottom element

Stack: list with (at least) two basic ops:

* Push an element onto its top
* Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

first	second	…	last

Queue: list with (at least) two basic ops:

* Append an element
* Remove first element

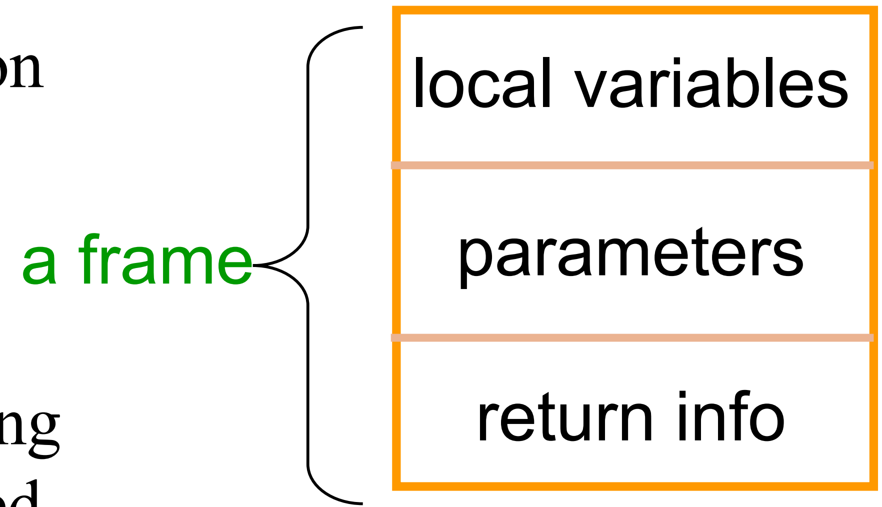
First-In-First-Out (FIFO)

Americans wait in a line. The Brits wait in a queue !

Stack Frame

A “frame” contains information about a method call:

At runtime Java maintains a stack that contains frames for all method calls that are being executed but have not completed.



Method call: push a frame for call on stack. Assign argument values to parameters. Execute method body. Use the frame for the call to reference local variables and parameters.

End of method call: pop its frame from the stack; if it is a function leave the return value on top of stack.

Memorize method call execution!

A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

To execute a method call:

1. push a frame for the call on the stack,
2. assign argument values to parameters,
3. execute method body,
4. pop frame for call from stack, and (for a function) push returned value on stack

When executing method body look in frame for call for parameters and local variables.

Frames for methods sum main method in the system


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
                    String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

frame: n ___
return info

frame: r ___ args ___
return info

Frame for method in the system that calls method main

frame: ?
return info

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
                    String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

Frame for method in the system that calls method main: main is then called

main	r ___ args ___ return info
system	? return info

Memorize method call execution!

To execute a method call:

1. push a frame for the call on the stack,
2. assign argument values to parameters,
3. execute method body,
4. pop frame for call from stack, and (for a function) push returned value on stack

The following slides step through execution of a recursive call to demo execution of a method call.

Here, we demo using: www.pythontutor.com/visualize.html

Caution: the frame shows not ALL local variables but only those whose scope has been entered and not left.

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
     if (n < 10) return n;
     return n%10 + sum(n/10);
}

public static void main(
       String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

Method main calls sum:

n 824
return info

main

r ___ args ___
return info

system

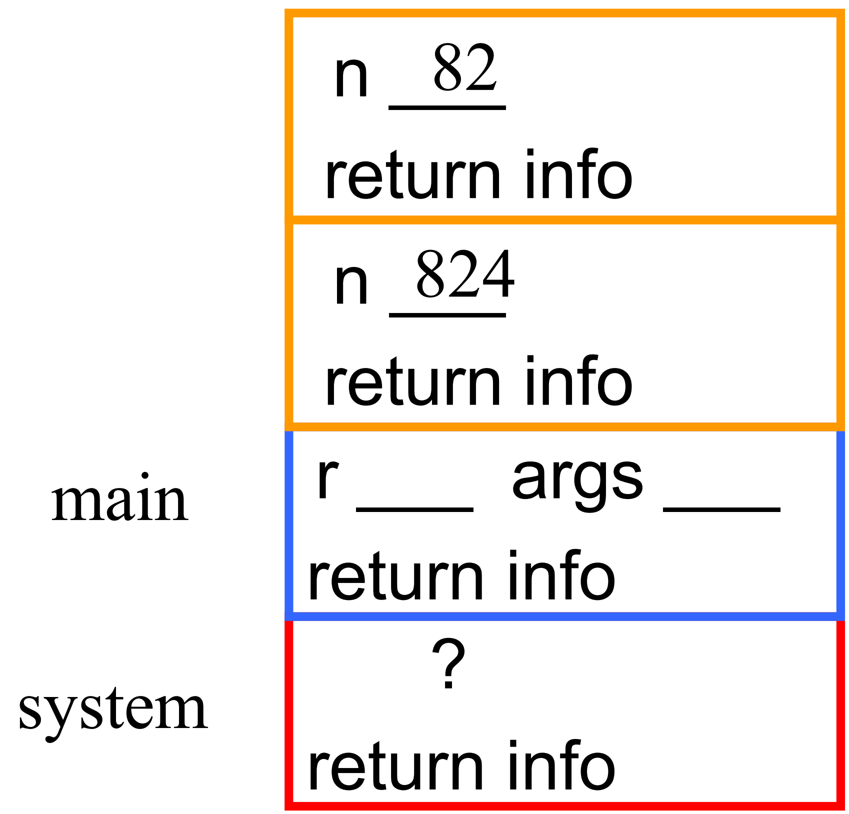
?
return info

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
       String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

n >= 10 sum calls sum:

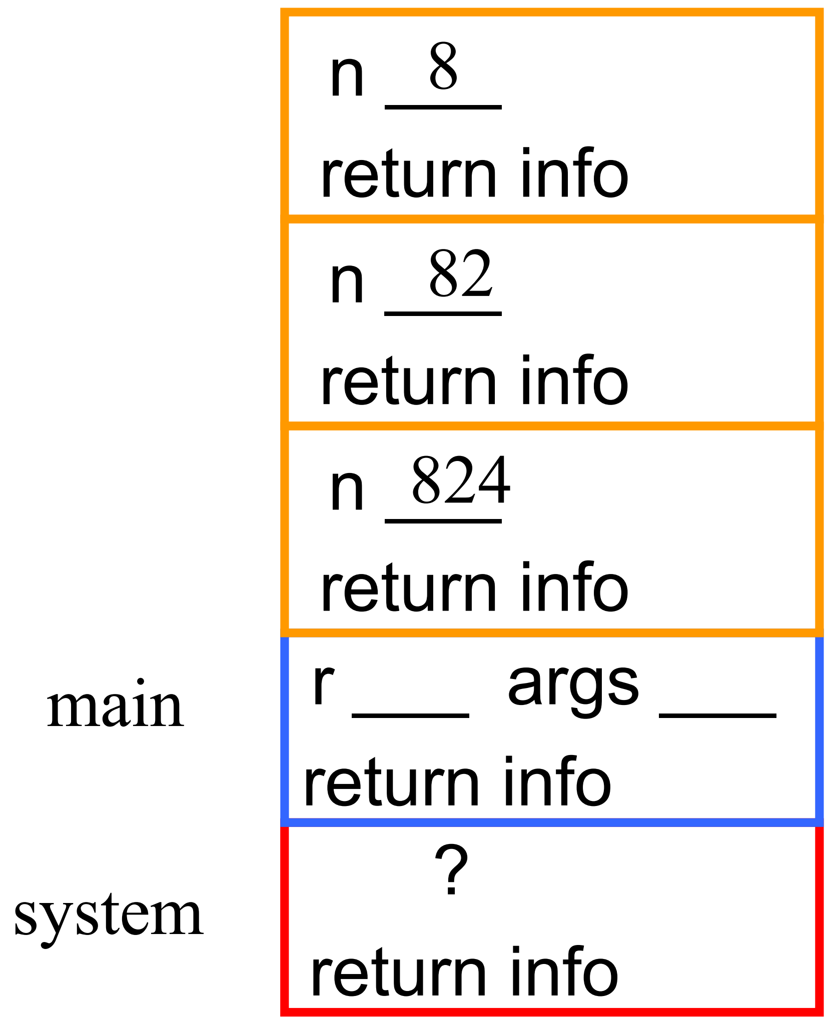


Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
      String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

n >= 10. sum calls sum:

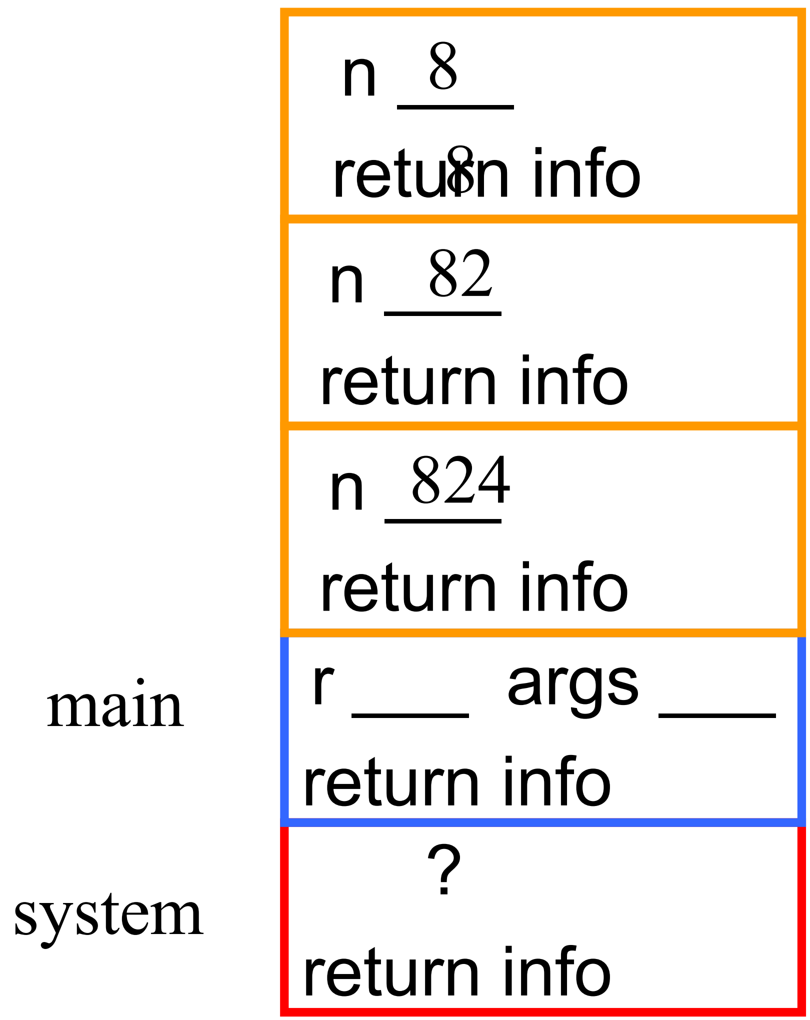


Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

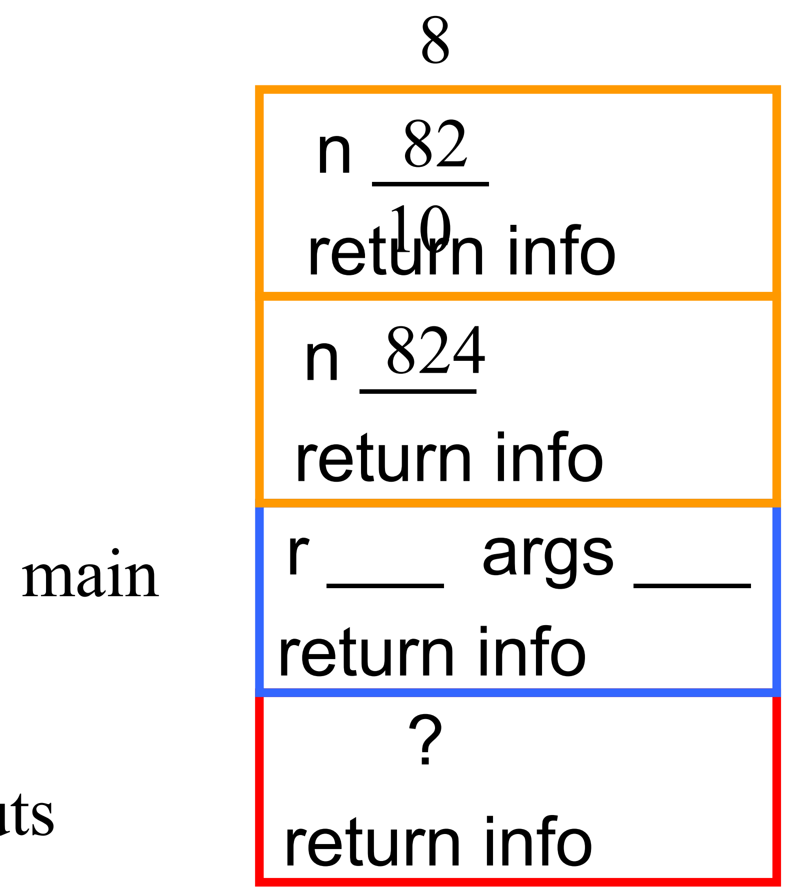
n < 10 sum stops: frame is popped and n is put on stack:



Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
      String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

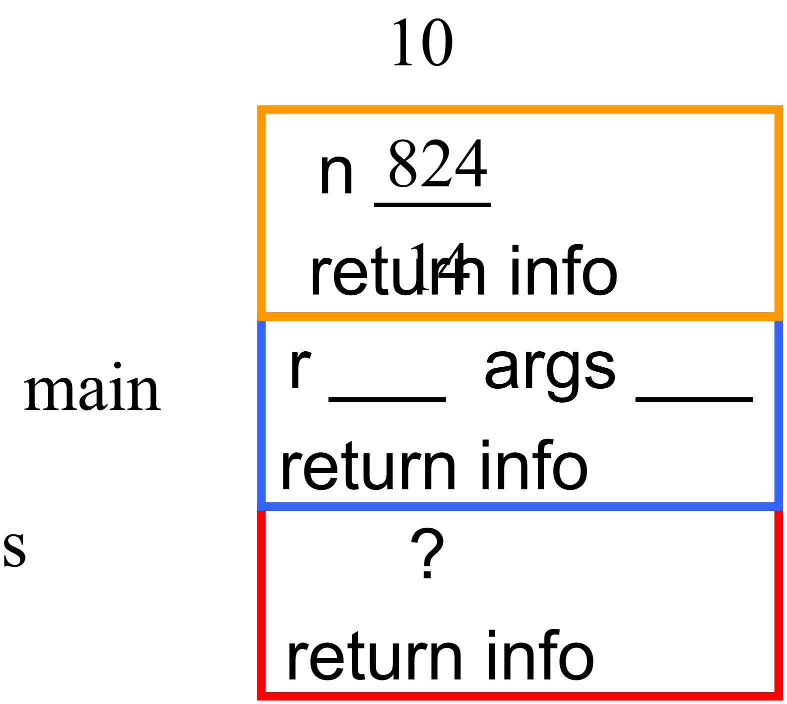


Using return value 8 stack computes
2 + 8 = 10 pops frame from stack puts
return value 10 on stack

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
      String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```



Using return value 10 stack computes
4 + 10 = 14 pops frame from stack
puts return value 14 on stack

Example: Sum the digits in a non-negative integer


```
public static int sum(int n) {
     if (n < 10) return n;
     return n%10 + sum(n/10);
}

public static void main(
       String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```

14

main

r 14   args __
return info

?
return info

Using return value 14 main stores 14 in r and removes 14 from stack

Poll time!

Two different questions, two different answers

1. How is it executed?

(or, why does this even work?)

It's **not** magic! Trace the code's execution using the method call algorithm, drawing the stack frames as you go.

Use only to gain understanding / assurance that recursion works.

2. How do we understand recursive methods?

(or, how do we write/develop recursive methods?)

This requires a totally different approach.

Back to Real Life Examples

Factorial function:

$0! = 1$

$n! = n * (n-1)!$ for $n > 0$

(e.g.: $4! = 4*3*2*1=24$)

Easy to make math definition into a Java function!

```java
public static int fact(int n) {
    if (n == 0) return 1;

    return n * fact(n-1);
}
```

Exponentiation:

$b^0 = 1$

$b^c = b * b^{c-1}$ for $c > 0$

```java
public static int exp(int b, int c) {
    if (c == 0) return 1;

    return b * exp(b, c-1);
}
```

How to understand what a call does

Make a copy of the method spec, replacing the parameters of the method by the arguments

spec says that the value of a call equals the sum of the digits of n

sumDigs(654)

sum of digits of **n**

sum of digits of **654**

```
/** = sum of the digits of n.
  * Precondition: n >= 0 */
public static int sumDigs(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return n%10 + sumDigs(n/10);
}
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): That is, cases where the parameter is small enough that the result can be computed simply and without recursive calls.

If n < 10 then n consists of a single digit.

Looking at the spec we see that that digit is the required sum.

```
/** = sum of the digits of n.
  * Precondition: n >= 0 */
public static int sumDigs(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return n%10 + sumDigs(n/10);
}
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

```
/** = sum of the digits of n.
  * Precondition: n >= 0 */
public static int sumDigs(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return n%10 + sumDigs(n/10);
}
```

```
return n%10 + sum(n/10);

return n%10 + (sum of digits of n/10);     // e.g. n = 843
```

Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does acc. to the spec and verify correctness.

```
/** = sum of the digits of n.
  * Precondition: n >= 0 */
public static int sumDigs(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return n%10 + sumDigs(n/10);
}
```

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

n/10 < n, so it will get smaller until it has one digit

Understanding a recursive method

Step 1. Have a precise spec!

Important! Can't do step 3 without precise spec.

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the spec and verify correctness.

Once you get the hang of it this is what makes recursion easy! This way of thinking is based on math induction which we don't cover in this course.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the parameters of the method

Writing a recursive method

Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed. Generally for “small” values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind. Done suitably, point 4 (about termination) is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the parameters of the method

Two different questions, two different answers

2. How do we understand recursive methods?

(or, how do we write/develop recursive methods?)

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the parameters of the method

Examples of writing recursive functions

For the rest of the class we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind.

Step 4. Make sure recursive calls are “smaller” (no infinite recursion).

Check palindrome-hood

A String palindrome is a String that reads the same backward and forward:

isPal("racecar") → true   isPal("pumpkin") → false

A String with at least two characters is a palindrome if

- (0) its first and last characters are equal and
- (1) chars between first & last form a palindrome:

have to be the same

e.g. AMANAPLANACANALPANAMA

have to be a palindrome

A recursive definition!

- A man a plan a caret a ban a myriad a sum a lac a liar a hoop a pint a catalpa a gas an oil a bird a yell a vat a caw a pax a wag a tax a nay a ram a cap a yam a gay a tsar a wall a car a luger a ward a bin a woman a vassal a wolf a tuna a nit a pall a fret a watt a bay a daub a tan a cab a datum a gall a hat a fag a zap a say a jaw a lay a wet a gallop a tug a trot a trap a tram a torr a caper a top a tonk a toll a ball a fair a sax a minim a tenor a bass a passer a capital a rut an amen a ted a cabal a tang a sun an ass a maw a sag a jam a dam a sub a salt an axon a sail an ad a wadi a radian a room a rood a rip a tad a pariah a revel a reel a reed a pool a plug a pin a peek a parabola a dog a pat a cud a nu a fan a pal a rum a nod an eta a lag an eel a batik a mug a mot a nap a maxim a mood a leek a grub a gob a gel a drab a citadel a total a cedar a tap a gag a rat a manor a bar a gal a cola a pap a yaw a tab a raj a gab a nag a pagan a bag a jar a bat a way a papa a local a gar a baron a mat a rag a gap a tar a decal a tot a led a tic a bard a leg a bog a burg a keel a doom a mix a map an atom a gum a kit a baleen a gala a ten a don a mural a pan a faun a ducat a pagoda a lob a rap a keep a nip a gulp a loop a deer a leer a lever a hair a pad a tapir a door a moor an aid a raid a wad an alias an ox an atlas a bus a madam a jag a saw a mass an anus a gnat a lab a cadet an em a natural a tip a caress a pass a baronet a minimax a sari a fall a ballot a knot a pot a rep a carrot a mart a part a tort a gut a poll a gateway a law a jay a sap a zag a fat a hall a gamut a dab a can a tabu a day a batt a waterfall a patina a nut a flow a lass a van a mow a nib a draw a regular a call a war a stay a gam a yap a cam a ray an ax a tag a wax a paw a cat a valley a drib a lion a saga a plat a catnip a pooh a rail a calamus a dairyman a bater a canal Panama

Example: Is a string a palindrome?


```
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;

    // { s has at least 2 chars }
    int n= s.length()-1;
    return s.charAt(0) == s.charAt(n)  &&  isPal(s.substring(1,n));
}
```

Substring from s[1] to s[n-1]

The Fibonacci Function

Mathematical definition:

$fib(0) = 0$
$fib(1) = 1$ two base cases!
$fib(n) = fib(n - 1) + fib(n - 2) \quad n \geq 2$

Fibonacci sequence: 0 1 1 2 3 5 8 13 …

```
/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
  if (n <= 1) return n;
  // { 1 < n }
  return fib(n-1) + fib(n-2);
}
```

Fibonacci (Leonardo Pisano) 1170-1240?

Statue in Pisa Italy
Giovanni Paganucci
1863

Example: Count the e's in a string


```
/** =  number of times c occurs in s */
public static int countEm(char c, String s) {
   if (s.length() == 0) return 0;

   // { s has at least 1 character }
   if (s.charAt(0) != c)
       return countEm(c, s.substring(1));

   // { first character of s is c}
   return 1 + countEm (c, s.substring(1));
}
```

substring s[1..]
i.e. s[1] …
s(s.length()-1)

- countEm('e', "it is easy to see that this has many e's") = 4
- countEm('e', "Mississippi") = 0