The invariant of Dijkstra's shortest-path algorithm
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The shortest-path algorithm is a breadth-first search algorithm. It has a loop, and each iteration of the loop will identify the shortest distance to one more node. We now present the invariant of the loop and prove a simple theorem about it.

The set of all nodes is partitioned into three sets: A *settled* set S (red), a *frontier* set F (blue), and a *far-off* set (black).

The invariant consists of three points:

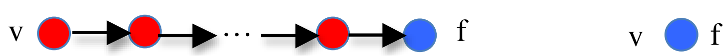
1. For a *settled* node s, at least one shortest path from v to s contains only settled nodes, and d[s] is the distance of that shortest path from v to s.

Think of the *settled* set as places that we know all about because we have visited often and settled there.

A node in the *frontier* has been visited at least once but not enough to know for certain that its shortest-path distance has been fully calculated. Think of the *frontier* as the moon and close planets that we know something about, but not everything. We have not settled there yet. Here's the second part of the invariant:

2. For a node f in the *frontier*, at least one path from v to f contains only settled nodes, except for the last one, f (as shown below), and d[f] is the minimum distance of all such paths.

That is, over paths that start with settled node v, perhaps contain more red nodes, and finally have one edge to f. There is a degenerate case: when f and v are the same node and it is in the *frontier* set. This degenerate case will make it easy to initialize the invariant, including putting v into the *frontier* set.



Here's the third part of the invariant:

3. All edges leaving the *settled* set end in the *frontier* set.

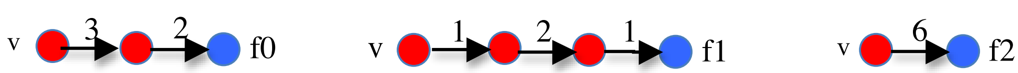
That's all there is to the invariant! Three simple and easy-to-remember points.

Stop the video at this point and convince yourself that if v is in either the *settled* or the *frontier* set, the invariant implies that d[v] = 0.

A theorem based on the invariant. We now prove an important theorem.

Theorem. For a node f in the *frontier* with minimum d value (over nodes in the *frontier*), d[f] is indeed the shortest-path distance from v to f.

For example, suppose the *frontier* contains three nodes f0, f1, and f2, with shortest-distances d[f0] = 5, d[f1] = 4, and d[f2] = 6. Then f1 is the node in the frontier with minimum d value.



We prove this theorem by showing that any other path from v to f does not have a smaller distance. We consider two cases:

Case 1. v and f are the same, and that node is in the *frontier* set. First, the settled set is empty, for, by invariant P1, if a node s is in the *settled* set than v must be there too, but it isn't; it is in the *frontier*. Second, since v is in the *frontier* set, by P2, the *frontier* set contains only one node, v. Third, we know that d[v] is 0, and 0 is the distance of the shortest path from v to v.

Case 2. v is in the *settled* set. Suppose that the shown path from v to f is the one with distance d[f].

By part 2 of the invariant, d[f] is the shortest-path distance over paths that contain only *settled* (red) nodes except for f. Consider any other v-f path. It starts at v, goes through *settled* nodes, visits f (so that path had distance ≥ d[f]) or vsits another *frontier* node g (say) and then winds its way to f. Since d[g] ≥ d[f], and since edge weights are positive, the distance of this path is ≥ d[f].

Q.E.D. (Quit-End-Done)